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Fermi Normal Coordinates and Conformal Fermi Coordinates in Cosmology

Master Thesis in Physics

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Abstract

The main aspect of this thesis is the construction and application of local inertial coordinates in Cosmology. According to the principle of general covariance upon which the Theory of General Relativity is built, the physical properties of a system are independent of the choice of coordinates employed to describe it. Local inertial coordinates are the natural frame that a freely falling observer builds in her proximity, in particular, Fermi Normal Coordinates (FNC) are the local inertial coordinates that the observer adopts along her world line. The metric in FNC is Minkowski plus quadratic corrections in the distance from the geodesics. In Cosmology, for a cosmological observer that is freely falling the patch can be studied with FNC is limited due to the existence of an intrinsic scale the Hubble scale. However, it is possible to build Conformal Fermi Coordinates (CFC) that preserve the advantage of FNC and has a larger patch of validity independent of Hubble. The metric in CNC is conformal Minkowski plus quadratic corrections in the distance from the geodesics. In this thesis, we construct both FNC and CFC and we work in linear order in perturbation theory around a RW metric.

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Contents

Part I
Introduction

Cosmology is the branch of science that studies the origin, evolution, and structure of the universe as a whole. It seeks to understand the fundamental principles that govern the universe on the largest scales, including its overall composition, expansion, and ultimate fate.

Cosmologists use a variety of observational and theoretical tools to study the nature of the universe. Observational techniques such as telescopes and satellites allow scientists to study the cosmic microwave background radiation, galaxies, clusters of galaxies, and other celestial objects. These observations provide valuable data that help shape our understanding of the history and structure of the Universe.

The theoretical framework of cosmology is based on Albert Einstein's theory of general relativity, which describes the gravitational interaction of matter and energy on cosmic scales. General relativity is a fundamental theory in physics that revolutionized our understanding of gravity. It was developed by Albert Einstein in the early 20th century as an extension of his special theory of relativity. General relativity proposes that gravity arises due to the curvature of space-time caused by mass and energy. According to this theory, massive objects such as stars and planets curve the fabric of space-time around them, creating what we perceive as gravity. In other words, gravity is not a force transmitted through space but rather a result of the geometry of space-time itself. General relativity is a profound theory that revolutionized our understanding of gravity, space-time, and the structure of the universe. This theory, together with the concept of the Big Bang, is the foundation of modern cosmology. The Big Bang Theory is a widely accepted scientific model that explains the origin and evolution of the universe. It suggests that the universe began as a singular point of extremely high density and temperature, and it has been expanding and evolving ever since. According to the Big Bang theory, around 13.8 billion years ago, all matter, energy, space, and time were compressed into an incredibly small and hot state, often referred to as a singularity. In an event known as the Big Bang, this singularity rapidly expanded, initiating the birth of the universe as we know it. In the early stages of the expansion, the universe was extremely hot and dense, and it underwent rapid and exponential expansion known as cosmic inflation. As the universe expanded, it cooled down, allowing fundamental particles such as protons, neutrons, and electrons to form. These particles eventually combined to form atoms, leading to the creation of matter. The Big Bang theory is supported by a wealth of observational evidence. One of the key pieces of evidence is the cosmic microwave background radiation, which is a faint glow of radiation left over from the early stages of the universe. This radiation, discovered in 1965, provides strong support for the idea that the universe was once in a hot and dense state. Additionally, observations of the red-shift of galaxies show that the universe is expanding. The further away a galaxy is, the greater its red-shift, indicating that galaxies are moving away from us and from each other.

Cosmologists also study the composition of the universe, which is thought to be made up mainly of dark matter and dark energy, in addition to ordinary matter. Dark matter is believed to make up about 85% of the matter in the universe, with the remaining 15% being ordinary matter. However, its exact nature and composition are still unknown. Dark matter, although invisible, exerts gravitational forces on visible matter and plays a crucial role in the formation of galaxies and large-scale structures. Dark energy is responsible for the accelerated expansion of the universe. It is a mysterious and hypothetical form of energy that is believed to permeate all of space and drive this accelerated expansion. The nature of dark energy remains largely unknown. One possible explanation is the presence of a cosmological constant. This concept was initially introduced by Albert Einstein in his theory of general relativity but later discarded when the universe was believed to be static. The accelerating expansion of the universe revived interest in the cosmological constant as a potential explanation for dark energy.

By studying the properties of the Universe and its constituents, cosmologists seek to answer fundamental questions about its origin, structure, and ultimate fate. Cosmology is a rapidly evolving field, and ongoing research continues to deepen our understanding of the cosmos, bringing us closer to unraveling the mysteries of the universe.

Contemporary cosmological models are based on the idea that the universe is almost the same everywhere but expands with time. Such a universe is described by the Robertson-Walker metric which is a fundamental tool in the field of cosmology that describes the geometry and dynamics of the expanding universe. The Robertson-Walker metric represents a four-dimensional space-time that describes the geometry of the universe. It is based on the assumption that the universe can be described as a homogeneous and isotropic space on large scales. Homogeneity implies that the properties of the universe are the same at every point, while isotopy means that it appears the same in all directions. The Robertson-Walker metric serves as the foundation for the Friedmann-Lemaître-Robertson-Walker (FLRW) model, which is widely used in modern cosmology. This model, combined with observations and theoretical principles, such as the Big Bang theory and the theory of general relativity, provides insights into the origin, expansion, and future fate of the universe. By utilizing the Robertson-Walker metric, cosmologists can study the behavior of galaxies, the cosmic microwave background radiation, the formation of large-scale structures, and other phenomena in the universe. It enables the exploration of fundamental questions about the nature of space, time, and the overall structure of our vast and evolving cosmos.

When we study cosmology, we need to perform calculations in a particular set of coordinates. Sometimes the calculation can be easier in one set of coordinates than in another. Therefore, the choice of an appropriate set of coordinates is very important in research. This is especially true in general relativity and its application to cosmological perturbation theory. A bad set of coordinates can greatly increase the computational complexity of the problem. So the choice of coordinates can be very critical.

In the context of general relativity, gravitation is not given rise by additional field propagating through space-time but is the curvature of space-time itself. The principle of equivalence indicates that for a local observer, one can always find a set of coordinates in which the metric takes the canonical form as in the flat space-time. Fermi normal coordinates (FNC) is such a locally inertial coordinates.

Choosing the natural set of coordinates of a local observer is very convenient because they are directly related to local measurements. In most applications to cosmology, the local observer is in an inertial frame, free-falling in the local gravitational potential. Such an inertial observer can describe the neighborhood around the time-like geodesic of the free-falling observer as a flat space-time with corrections that grow with the square of the distance from the observer's geodesic times the second derivatives of the space-time metric which is described by the Riemann tensor. This holds in any space-time, and the set of coordinates is known as Fermi Normal Coordinates (FNC).

However, when we apply the FNC to Cosmology, due to the expanding universe the patch that can be covered by FNC is limited. For this reason, the conformal Fermi coordinates (CFC) are established to solve the problem. Which will be discussed in this thesis.

Another thing we are going to talk about is that the real universe is not perfectly homogeneous and isotropic. These little perturbations are very important to the studying of the evolution of the universe. Perturbation theory is a mathematical framework used to study the evolution and behavior of small deviations or fluctuations from the isotropic and homogeneous background of the universe. It provides a powerful tool for understanding how these perturbations grow and evolve over time, ultimately leading to the formation of galaxies, galaxy clusters, and other large-scale structures in the universe. To conclude, Cosmology aims to explain the origin and evolution of the entire contents of the Universe, the underlying physical processes, and thus to gain a deeper understanding of the physical laws that are assumed to apply throughout the Universe.

Part II
Gravity is Geometry

In the context of general relativity, gravitation is not given rise by an additional field propagating through space-time but is the curvature of space-time itself. This is Einstein's profound insight and this idea is based on the Principle of Equivalence. This physical principle leads us to describe gravity as the geometry of a curved manifold in which the curvature of space-time is described by the Riemann tensor.

1 Principle of Equivalence

1.1 Weak Equivalence Principle

In Newtonian mechanics, the second law relates the force on an object and the acceleration of the object by a proportional relation

$$\mathbf{F} = m_i \mathbf{a}. \tag{1}$$

The constant m_i is defined as the inertial mass. In other words, the inertial mass is a constant that relates to the resistance you feel when pushing the object and this constant has a universal character which means the value is not relevant to the type of force is being exerted. Newton's law of gravitation also states that the gravitation force exerted on an object is proportional to the gradient of the gravitational potential Φ which is a scalar field

$$\mathbf{F}_g = -m_g \nabla \Phi. \tag{2}$$

Where m_g is another constant defined as the gravitational mass. At first sight, we have no reason to say that the inertial mass m_i should be the same as the gravitational mass m_g. Nevertheless, experiments have shown that every object in a gravitational field falls at the same rate which means the response of matter to gravity is universal. This leads us to the idea of the Weak Equivalence Principle (WEP)[1], which simply states that

$$m_i = m_g \tag{3}$$

for any object with any composition. We easily get that for objects that are only gravitationally attracted which we will call "free-falling" objects the acceleration is

$$\mathbf{a} = -\nabla \Phi. \tag{4}$$

It shows that the behavior of a free-falling test particle is independent of its mass or any other quality of it.

In the language of "space-time" and "trajectory" this statement is translated into that there exists a preferred class of trajectories through space-time on which unaccelerated particles travel. Here "unaccelerated" means the particles are subjected only to gravity, so the trajectories are called "free-falling" trajectories. That is to say, if an uncharged test body is placed at an initial event in space-time and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition. Where "uncharged" means an electrically neutral body that has negligible self-gravitational energy and that is small enough in size so that its coupling to inhomogeneities in external fields can be ignored.

The WEP can also be stated in another form. Imagine particles in a small box, by observing the behavior of the particles there is no way to tell if these particles are in a gravitational field or the box is accelerating at a constant rate. But if the box is too big then the gravitational field would change from place to place in an observable way, while the effect of the acceleration will be the same everywhere in the box. The WEP can therefore be stated as: The motion of freely-falling particles is the same in a gravitational field and a uniformly accelerated frame, in small enough regions of space-time.

1.2 Einstein Equivalence Principle and Strong Equivalence Principle

In the WEP we know that there is no way for the observer in the box to distinguish between external gravitational field and uniform acceleration by simply dropping the particles. The WEP alone does not highly constrain the dynamic of the theory of gravitation. In fact, there are many theories other than the General Relativity that satisfied the WEP.[2] There exist additional physical principles that make GR stand out: the Einstein equivalence principle (EEP) and in particular the strong equivalence principle (SEP).[3][4]

The idea of EEP is simple, not by the motion of the particles, but by any local non-gravitational experiment the observer cannot distinguish between an external gravitational field and a uniform acceleration. The Einstein Equivalence Principle then states: (i) WEP is valid, (ii) the outcome of any local non-gravitational test experiment is independent of the velocity of the freely falling apparatus, and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed. This principle is at the heart of gravitation theory, for it is possible to argue convincingly that if EEP is valid, then gravitation must be a curved-space-time phenomenon.

The Strong Equivalence Principle is a more inclusive principle. The idea of SEP is that in small enough regions of space-time, it is impossible to detect the existence of a gravitational field by means of any local experiments including both gravitational experiments and non-gravitational experiments. SEP states that (i) WEP is valid for self-gravitating bodies as well as for test bodies, (ii) the outcome of any local test experiment, gravitational or non-gravitational, is independent of the velocity of the freely falling apparatus, and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed.

Einstein's theory of general relativity is thought to be the only theory of gravity that satisfies the strong equivalence principle.[3]

2 Curvature of Space-time and Gravitation

The EEP implies that there are no such things that are "gravitational neutral objects" since no experiment can tell the acceleration and gravitational field apart. The fact that the gravitational neutral object does not exist means gravity is inescapable. Considering this fact, it makes more sense to define "unaccelerated" as "free-falling". In other words, an object subjected only to gravitational force is considered zero acceleration under our definition. From here we are led to the idea that gravity is not a force since force is something that leads to acceleration. This suggests that we should attribute the action of gravity to the curvature of the space-time. But the EEP indicates that in a small enough region the space-time is flat. We, therefore, describe space-time with a kind of mathematical structure that looks locally like Minkowski space but may possess nontrivial curvature over extended regions. This mathematical structure is a very important conception of mathematics and physics which is known as the manifold. The geometry of the manifold is defined by the metric so it is clear that the curvature is related to the metric but it is not immediately clear how to relate the curvature to the metric since the metric depends on the coordinate system. But we will formalize the curvature into a mathematical structure from the metric.[5][1]

2.1 Tensor fields

A vector is a perfectly well-defined geometric object. A real vector space is a collection of vectors that can be added and multiplied by real numbers in a linear way:

$$(a+b)(A+B) = aA + aB + bA + bB \tag{5}$$

where a, b are real numbers and A, B are vectors. Any vector can be written as a linear combination of basis vectors. If A is a vector and $\hat{e}_{(\mu)}$ is a set of basis, i.e. a set of vectors that not only span the vector space but also are linearly independent, the vector

$$A = A^{\mu}\hat{e}_{(\mu)} \tag{6}$$

Where A^{μ} are the components of the vector.

A dual vector space is the space of all linear maps from a vector space to the real numbers, such that

$$\omega(aA + bB) = a\omega(A) + b\omega(B) \in \mathbf{R} \tag{7}$$

where ω is a dual vector, A, B are vectors and a, b are real numbers. The basis of the dual vector space can be constructed by demanding

$$\hat{\theta}^{(\nu)}\left(\hat{e}_{(\mu)}\right) = \delta^{\nu}_{\mu}. \tag{8}$$

A dual vector thus can be written as

$$\omega = \omega_{\mu}\hat{\theta}^{(\mu)} \tag{9}$$

where ω_{μ} are the components of the dual vector. The action of a dual vector field on a vector field is a scalar field:

$$\begin{aligned} \omega(V) &= \omega_{\mu}\hat{\theta}^{(\mu)}\left(V^{\nu}\hat{e}_{(\nu)}\right) \\ &= \omega_{\mu}V^{\nu}\hat{\theta}^{(\mu)}\left(\hat{e}_{(\nu)}\right) \\ &= \omega_{\mu}V^{\nu}\delta^{\mu}_{\nu} \\ &= \omega_{\mu}V^{\mu} \in \mathbf{R}. \end{aligned} \tag{10}$$

A straightforward generalization of vectors and dual vectors is the notion of a tensor. A dual vector is a linear map from vectors to $\mathbf{R}$ while a tensor of rank (k, l) is a multi-linear map from a collection of dual vectors and vectors to $\mathbf{R}$. From this point of view, a scalar is a rank $(0, 0)$ tensor, a vector is a rank $(1, 0)$ tensor and a dual vector is a rank $(0, 1)$ tensor. The tensor product between a (k, l) tensor T and a (m, n) tensor S gives a $(k+m, l+n)$ tensor

$$\begin{aligned} &T \otimes S\left(\omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)}\right) \\ =&T\left(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}\right) \\ &\times S\left(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots, V^{(l+n)}\right). \end{aligned} \tag{11}$$

A basis for the space of all $(k+m, l+n)$ is then

$$\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(v_l)}. \tag{12}$$

Thus, we are able to write a (k, l) tensor as

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots v_l} \hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(v_1)} \otimes \cdots \otimes \hat{\theta}^{(v_l)} \tag{13}$$

where $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots v_l}$ are the components of the tensor.

The set of all vectors at a point P on a manifold is the tangent space T_P. We choose the coordinate basis for the tangent space T_P as the basis. Coordinate basis is represented by the partials $\partial_\mu = \hat{e}_{(\mu)}$. If we consider a coordinate transformation between two coordinate systems $x^\mu \to x'^{\mu'}$, the chain rule gives

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu. \tag{14}$$

The vector is unchanged under the change of the basis

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu, \end{aligned} \tag{15}$$

hence,

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \tag{16}$$

is the vector transformation law. Further, the gradients dx^μ are a basis of dual vectors since

$$\mathrm{d}x^\mu \left(\partial_v\right) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_v. \tag{17}$$

If we do the coordinates transformation $x^\mu \to x'^{\mu'}$, we find

$$\mathrm{d}x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \mathrm{d}x^\mu. \tag{18}$$

and thus the transformation law of the dual vector is

$$\omega_{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \omega_\mu. \tag{19}$$

In the end, we find that under the coordinate basis ∂_μ and d^μ for vector and dual vector respectively, the coordinates transformation law of a tensor is[1][6]

$$T^{\mu_1' \cdots \mu_k'}{}_{v_1' \cdots v_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \cdots \mu_k}{}_{v_1 \cdots \nu_l}. \tag{20}$$

2.2 Covariant derivative and Christoffel symbols

In flat space-time in inertial coordinates, the partial derivative operator ∂_μ is a map from (k, l) tensor fields to $(k, l+1)$ tensor fields. But the map of the partial derivative depends on the coordinate system used since the partial derivative is not a good tensor operator. For this reason, we want to define an operator to perform the functions of the partial derivative which should be a map from (k, l) tensor fields to $(k, l+1)$ tensor fields but in a way independent of coordinates.

By requiring the properties of linearity and the Leibniz rule we define a covariant derivative of a vector as

$$\nabla_\mu V^\upsilon = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{21}$$

Where Γ is the connection coefficient. It is clear that the connection coefficient is not a tensor because it is constructed to be nontensorial but is constructed to make the combination with partial derivative to be able to transform as a tensor.[1]

Similarly, we can also express the covariant derivative of a one-form as a partial derivative plus some linear transformation.

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \widetilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda \tag{22}$$

Consider now the properties of commuting with contractions and reducing to the partial derivative on scalars and we have

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\nu_{\mu\lambda} \tag{23}$$

Up to now we still have not completely defined the connection coefficient yet. Therefore we want to introduce two additional properties to define the unique connection coefficient: torsion-free

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)} \tag{24}$$

and metric compatibility

$$\nabla_\rho g_{\mu\nu} = 0. \tag{25}$$

Where torsion tensor is defined by

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}. \tag{26}$$

A connection is metric compatible if the covariant derivative of the metric with respect to that connection is everywhere zero and a metric-compatible covariant derivative commutes with the raising and lowering of indices. With these properties, we demand the formula of the connection coefficient be defined

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \tag{27}$$

This connection we derive from the metric is also known as the Christoffel connection and the connection coefficient is called Christoffel symbol.[1][3]

2.3 Parallel transport, the geodesic equation and the Riemann curvature tensor

We think of a derivative as a way of quantifying how fast something is changing. Actually, the covariant derivative quantifies the instantaneous rate of change of a tensor field in comparison to what the tensor would be if it were "parallel transported". Where "parallel transport" means keeping the vector constant when moving it along a path. However, there is a crucial difference between flat and curved spaces, in a curved space, the result of transporting a vector from one point to another will depend on the path taken between the points so it appears as if there is no natural way to uniquely move a vector from one tangent space to another. Actually, this indicates that two vectors can only be compared in a natural way if they are elements of the same tangent space. Therefore what we can do is to define the directional covariant derivative. Given a curve $x^\mu(\lambda)$, the directional covariant derivative is defined to be the covariant derivative along the curve

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu. \tag{28}$$

This is a map defined only along the path, from (k,l) tensors to (k,l) tensors. Parallel transport of the tensor T along the path $x^\mu(\lambda)$ is defined to be that the covariant derivative of T along the path vanished. For a vector, the equation of parallel transport takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0. \tag{29}$$

After the introduction of the parallel transport, we are now interested in defining the straight line: a straight line is a path that parallel transports its own tangent vector. The tangent vector to a path $x^\mu(\lambda)$ is $dx^\mu/d\lambda$. The condition for the tangent vector to be parallel transported is thus

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0, \tag{30}$$

or

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0. \tag{31}$$

This equation is also known as the geodesic equation. We can easily check that in Euclidean space and Cartesian coordinates $\Gamma^\mu_{\rho\sigma} = 0$ so the geodesic equation is simply $d^2x^\mu/d\lambda^2 = 0$, which is the equation for a straight line.

Now we are finally prepared to define the Riemann tensor which quantified the curvature. We have mentioned that the result of transporting a vector from one point to another will depend on the path taken between the points in non-flat space-time. Therefore the result of transporting reflects the total curvature enclosed by the path. Riemann tensor is introduced by considering parallel transport around an infinitesimal loop. As discussed before the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it was parallel transported. Therefore the commutator of two covariant derivatives in two directions measures the difference between parallel transporting a tensor first one way and then another, versus in the opposite order of the two directions. The calculation is straightforward.

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] V^\rho &= \left(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\upsilon\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right)V^\sigma - 2\Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\rho \\ &= R^\rho_{\sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}\nabla_\lambda V^\rho\end{aligned} \tag{32}$$

Where the Riemann tensor is identified as[7]

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\upsilon\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{33}$$

From this expression it is easy to find the properties of the Riemann tensor: The Antisymmetry of the first two indexes

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \tag{34}$$

the Antisymmetry of the last two indices

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \tag{35}$$

the invariance of the exchange of the first and the second two indices

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \tag{36}$$

and the vanishing of the sum of cyclic permutations of the last three indices

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0. \tag{37}$$

The Riemann tensor may be contracted to give the Ricci tensor

$$R_{\mu\nu} = g^{\lambda\rho} R_{\rho\mu\lambda\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{38}$$

Because of the symmetry of the Riemann tensor the Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu}. \tag{39}$$

The trace of the Ricci tensor in the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}. \tag{40}$$

Now we have found the curvature tensor which defines the geometry of the manifold and the relationships between different components of the Riemann tensor. Considering the antisymmetry, symmetry, and cyclicity properties the number of independent components of the Riemann tensor is left with

$$\frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) - \frac{1}{24}n(n-1)(n-2)(n-3) = \frac{1}{12}n^2(n^2-1). \tag{41}$$

For $n = 1$ the number of the independent components of the Riemann tensor is 0. Actually, in one dimension of space-time, the Riemann tensor always vanishes. This should remind us that the Riemann tensor reflects only the inner properties of the space-time, not how it is embedded in a higher dimensional space-time. The Riemann tensor measures the intrinsic geometry of a space-time which can be measured by observers confined to the manifold.[1]

2.4 Einstein's Equation

Einstein's equation governs how the metric responds to energy and momentum. According to the conservation of the energy and momentum the energy-momentum tensor $T_{\mu\nu}$ should obey

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{42}$$

The Einstein tensor defined as[1]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{43}$$

is a symmetric $(0, 2)$ tensor, constructed from the Ricci tensor, which is automatically conserved because it satisfies the Bianchi identities. Thus, we relate the Einstein tensor to the energy-momentum tensor as

$$G_{\mu\nu} = kT_{\mu\nu} \tag{44}$$

By contracting both sides we have $R = -kT$ using which we rewrite the equation as

$$R_{\mu\nu} = k(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \tag{45}$$

Considering a perfect-fluid source and going to the Newtonian limit we find $k = 8\pi G$ where G is Newton's constant of gravitation. Thus, we can present Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{46}$$

The most general expression of Einstein's equation includes an additional term proportional to the metric tensor that is allowed by the metric compatibility.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{47}$$

Where Λ is the cosmological constant which is the main candidate for dark energy in the current standard model of cosmology.[8] This equation tells how the curvature of the space-time reacts to the presence of the energy-momentum. In vacuum where $T_{\mu\nu} = 0$ the vacuum Einstein equation is simply $R_{\mu\nu} = 0$.

3 Locally Inertial Coordinates

The Einstein equivalence principle indicates that in a small enough region the space-time is flat. Actually, for a point P on a manifold it should always exist a coordinate system $\hat{x}^\mu$ in which $\hat{g}_{\mu\nu}$ takes the canonical form and the first derivatives $\hat{\partial}_\sigma \hat{g}_{\mu\nu}$ vanishes. Where the canonical form means that in this form the metric components become

$$\hat{g}_{\mu\nu} = diag(-1, ..., 1, ...0, ...). \tag{48}$$

The idea to prove this is to consider the transformation of the metric

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}} \frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}} g_{\mu\nu}, \tag{49}$$

and Taylor expand both sides

$$LHS = (\hat{g}_{\hat{\mu}\hat{\nu}})_p + (\hat{\partial}_{\hat{\mu}_1} \hat{g}_{\hat{\mu}\hat{\nu}})_p \hat{x}^{\hat{\mu}_1} + \mathcal{O}(\hat{x}^2) \tag{50}$$

RHS,

$$\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}} = \left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\right)_p + \left(\frac{\partial^2 x^\mu}{\partial \hat{x}^{\hat{\mu}_1} \partial \hat{x}^{\hat{\mu}}}\right)_p \hat{x}^{\hat{\mu}_1} + \mathcal{O}(\hat{x}^2) \tag{51}$$

$$g_{\mu\nu} = (g_{\mu\nu})_p + (\hat{\partial}_{\hat{\mu}} g_{\mu\nu})_p \hat{x}^{\hat{\mu}} + \mathcal{O}(\hat{x}^2) \tag{52}$$

Set terms of equal order in $\hat{x}$ on each side equal

$$\begin{aligned}(\hat{g}_{\hat{\mu}\hat{\nu}})_p + (\hat{\partial}_{\hat{\mu}_1} \hat{g}_{\hat{\mu}\hat{\nu}})_p \hat{x}^{\hat{\mu}_1} = &\left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\right)_p \left(\frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}}\right)_p (g_{\mu\nu})_p + \left[\left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\right)_p \left(\frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}}\right)_p (\hat{\partial}_{\hat{\mu}} g_{\mu\nu})_p \right.\\ &\left. + \left(\frac{\partial x^\nu}{\partial \hat{x}^{\hat{\nu}}}\right)_p \left(\frac{\partial^2 x^\mu}{\partial \hat{x}^{\hat{\mu}_1} \partial \hat{x}^{\hat{\mu}}}\right)_p (g_{\mu\nu})_p + \left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\right)_p \left(\frac{\partial^2 x^\nu}{\partial \hat{x}^{\hat{\mu}_1} \partial \hat{x}^{\hat{\nu}}}\right)_p (g_{\mu\nu})_p\right] \hat{x}^{\hat{\mu}_1}\end{aligned} \tag{53}$$

Consider a 4-dimension spacetime. For the zeroth order, $(\hat{g}_{\hat{\mu}\hat{\nu}})_p$ has 10 independent components while $\left(\frac{\partial x^\mu}{\partial \hat{x}^{\hat{\mu}}}\right)_p$ has $4 \times 4 = 16$ independent components. Therefore we are free to choose 16 components which is more than enough to put 10 components of the metric into canonical form. For the first order, the matrix $(\hat{\partial}_{\hat{\mu}_1} \hat{g}_{\hat{\mu}\hat{\nu}})_p$ has $4 \times 10 = 40$ independent components and the matrix $\left(\frac{\partial^2 x^\mu}{\partial \hat{x}^{\hat{\mu}_1} \partial \hat{x}^{\hat{\mu}}}\right)_p$ has $4 \times 10 = 40$ degrees of freedom. This is enough freedom for us to set the first derivative of the metric to zero. For the second order, it is not possible to set the second derivative of the metric to zero. Such coordinates are called locally inertial coordinates. In locally inertial coordinates the metric at p looks like the metric of the Minkowski space-time to first order.[1]

$$\hat{g}_{\mu\nu}(p) = \eta_{\mu\nu}, \quad \hat{\partial} \hat{g}_{\mu\nu}(p) = 0 \tag{54}$$

Part III

The Geometry of Our Universe

4 The Standard Model of Cosmology

The Standard Model of Cosmology, also known as the ΛCDM Model or Concordance Model, is the comprehensive scientific framework with which we understand our universe. In the late

1920s, the first cosmological observations started to take place, and immediately our philosophical prejudice on the universe had to be abandoned. The ever-increasing amount of precise astrophysical and cosmological data revealed unexpected results like the existence of unknown energy components, together with a mysterious background of radiation that has the same temperature in every direction. In the late 1990s, the ΛCDM model emerged as a coherent physical description of our universe[9] and benchmarked the birth of Modern Cosmology as an actual observational science.

The Standard Model of Cosmology is based on the modern formulation of the Copernican Principle, which goes by the name of Cosmological Principle, and states that in the universe there is no preferred position or direction.[10] In other words, the Cosmological Principle asserts that the universe is homogeneous and isotropic on sufficiently large cosmological scales. Another assumption of the ΛCDM Model is that Einstein's theory of General Relativity is the correct theory of gravity on cosmological scales, and is the main interaction that drives the evolution of the universe. Both assumptions have been well-tested experimentally and are at the core of Modern Cosmology.

To summarize the concordance model of cosmology: a Euclidean universe that is dominated today by non-baryonic cold dark matter (CDM) and a cosmological constant Λ, with initial perturbations generated by inflation in the very early universe.[11]

4.1 Hubble's Law

The first compelling cosmological data are attributed to the astronomer Edwin Hubble, who in 1929 measured the distance and red-shift of nearby galaxies. Hubble observed that the spectra of these galaxies appeared redder than expected and concluded that they were receding from the Milky Way. Moreover, he noticed that the further a galaxy is from us, the fastest it is receding away. Hubble's fundamental discovery was that the recession velocities V of the distant galaxies increased linearly with distance D.

$$V = cz = H_0 D \tag{55}$$

This direct proportionality between receding velocity and distance is known as Hubble's law and the proportionality constant is the Hubble constant H_0. At first glance, it seems that there was a great explosion in our neighborhood from which the galaxies are thrown out. However, we should not consider ourselves in a special position in the universe so in another position in the universe we should be able to observe the same phenomenon. In fact, the isotropic and homogeneous of the universe suggests a highly regular structure of the universe with no preferred direction. Therefore the galaxies might be just points embedded in a space that is expanding and indeed the idea of contemporary cosmology is that the universe is expanding.

The distance at which objects are receding from us at the speed of light due to the overall expansion of the universe is the Hubble radius $R_H = \frac{c}{H_0}$. The Hubble radius is a theoretical construct used in cosmology to describe the size of the observable universe at a given time. It is not a physical boundary but rather a characteristic scale associated with the expansion of space. Objects beyond the Hubble radius are receding from us faster than the speed of light due to the overall expansion, making them unobservable to us.[12]

4.2 Thermal History of the Universe and the CMB

In the early Universe, the Universe was denser and hotter, dominated by relativistic particles and radiation. Because of its high energy, particles and anti-particle pairs are created and annihilated.[13] In the strictest mathematical sense it is not possible for the Universe to be in thermal equilibrium, as the FRW cosmological model does not possess a time-like Killing vector. For practical purposes, however, the Universe has for much of its history been very nearly in thermal equilibrium. The key to understanding the thermal history of the Universe is the comparison of the particle interaction rates and the expansion rate.[14]

At $T \gg 1TeV$, inflationary expansion and baryogenesis take place. During this stage, quarks and gluons are not bound to hadronic states, such that there exist no protons, neutrons, and so on. The Universe was made of fundamental elementary particles, forming a hot plasma. At $T \approx 150MeV$, the quark–hadron phase transition occurs, confining quarks into hadrons. At $T \approx 100MeV$, pions become non-relativistic and begin to annihilate. From this point on, protons and neutrons are the only hadronic species left. At $T \approx 1MeV$, electrons and positrons become non-relativistic, annihilating each other. The weak interactions become ineffective, and the ratio of neutrons to protons is frozen. At $T \approx 0.1MeV$, the Big Bang Nucleosynthesis (BBN) starts, synthesizing protons and neutrons to produce D, He, and a few other heavy elements. This nuclear fusion is exactly the same as one at the core of stars, but it takes place everywhere in the Universe. $T \approx 3000K$, free electrons and protons recombine to form neutral hydrogen atoms. The Universe then becomes transparent to photons, and these free-streaming photons are observed today as the cosmic microwave background (CMB) in a black-body distribution. [15][16]

In 1965, Arno Penzias and Robert Wilson discovered an isotropic background of microwave radiation using a microwave antenna at Bell Labs.[17] This microwave background is known as the Cosmic Microwave Background (CMB). More recently, the Cosmic Background Explorer (COBE) satellite has revealed that the Cosmic Microwave Background is exquisitely well fitted by a blackbody spectrum with a temperature $T_0 = 2.725K$. The existence of the CMB is a very important cosmological clue that the universe has a thermal origin. The existence of blackbody radiation can be explained by an expanding universe that in the past was in a hot and dense plasma phase.

We have mentioned that the CMB temperature was about $3000K$ and the temperature of the CMB today is $2.725K$, a factor of 1100 lower. The drop in temperature of the blackbody radiation is a direct consequence of the expansion of the universe.[18][19]

4.3 Cold Dark Matter and Cosmological Constant

The discovery of the type Ia supernova 1997ff [20] marked the beginning of a new era in cosmology and physics. The analysis of the emission of this type of supernova led to the discovery that our universe is in a state of accelerated expansion. This is not natural to understand because gravity, as we know it, should attract matter, slowing down the expansion. One possibility is that there is a new form of matter or rather energy, that acts as an anti-gravitational force. This is now widely known as Dark Energy (DE), and its nature is still a mystery to us. The simplest and most successful candidate for DE is the cosmological constant Λ.

Another dark part of our universe is called dark matter. Many observations of different nature and from different sources at different distance scales point out the existence of the dark matter. One of these observations is the weak lensing. X-ray maps made by the combination of X-ray and weak lensing observational techniques [21] show the result of a merging between the hot gases of two galaxy clusters which gravitational lensing maps reveal to be lagging behind their respective centers of mass. Therefore, most parts of the clusters simply went through one another, leaving behind a smaller fraction of hot gas. This is seen as direct empirical proof of the existence of dark matter forming a massive halo and a gravitational potential well in which gas and galaxies lie. The observational evidence for dark matter suggests not only that it exists, but also that it must have negligible pressure or, equivalently, a small velocity of its particles, much less than the speed of light. Hence Cold Dark Matter (CDM).

The combined observational successes of Λ and CDM form the so-called ΛCDM model, which is the standard model of cosmology.[22]

5 Homogeneous and Isotropic Universe

Contemporary cosmological models are based on the idea that the universe is almost the same everywhere but expands with time which can be related to a manifold that is spatially homogeneous and isotropic but evolving in time. Therefore in general relativity this translates into the statement that the universe can be foliated into spacelike slices such that each three-dimensional slice is maximally symmetric. Thus we consider our space-time to be $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ represents the time direction and Σ is a maximally symmetric three-manifold. According to the astronomical observation the universe is homogeneous and isotropic on the scales of about 10^{18} and larger. If we take the solar system as example the idea that the universe is homogeneous and isotropic may sounds impossible but on a larger scale it is reasonable to say that. Because on very large scale the local density variations are averaged over. There are a few observational evidences for the homogeneity and isotropy of the universe such as the distribution of galaxies; the isotropy of the distribution of radio sources on the sky and the remarkable isotropy of the cosmic microwave background (CMB).

5.1 Homogeneity and Isotropy: Maximally Symmetric Space

The homogeneity of a manifold means that the metric is the same throughout the manifold. Formally it states that given any two points p and q in the manifold, there is an isometry that takes p into q.

Isotropy applies to specific points in the manifold. It states that the space looks the same in any direction. The formal definition is that a manifold is isotropic around a point p if, for any two vectors V and W in the tangent space of the manifold, there is an isometry of the manifold such that the pushforward of W under the isometry is parallel with V.

Note that there is no relevance between isotropic and homogeneous. A manifold can be homogeneous but not isotropic around any point or it can be isotropic around a certain point but not homogeneous. If the manifold is isotropic around a point and it is also homogeneous then the manifold is isotropic everywhere, thus, homogeneous and isotropic. The observation of the remarkable isotropy indicates the isotropy of the universe. The Copernican principle states that humans, on the Earth or in the Solar System, are not privileged observers of the universe. In other words, there is no reason to believe that we are the center of the universe so what we observe should be the same everywhere in the universe. Therefore we assume that the universe is isotropic and homogeneous.

Isotropy indicates that the space is invariant under rotations and homogeneity implies that the space is invariant under translations. Considering an isotropic and homogeneous space of n dimensions, the total number of independent translations is simply n and the total number of independent rotations is $\frac{1}{2}n(n-1)$ since there are n axes and for each axis it can rotate in $n-1$ other axes, but one axis rotating towards another is the same as the reverse. Therefore, an n dimensions isotropic and homogeneous space $\mathbf{R}^n$ has $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ independent symmetries which also implies that the space $\mathbf{R}^n$ has $\frac{1}{2}n(n+1)$ independent Killing vector fields. Where Killing vector fields are the vector fields that satisfy the Killing's equation

$$\nabla_{(\mu} K_{\nu)} = 0. \tag{56}$$

Every Killing vector implies the existence of conserved quantities associated with geodesic motion. If there is a vector that satisfies the Killing's equation it is always possible to find a coordinate system in which the metric is independent of some coordinate. Physically it means the metric is preserved along the direction of the Killing vector. We refer to an n dimensions space $\mathbf{R}$ with $\frac{1}{2}n(n+1)$ independent Killing vector fields as a maximally symmetric space.[1]

Therefore the isotropy and homogeneity of the universe imply that the space of the universe is maximally symmetric which also implies that the space of the universe has the maximum possible number of Killing vectors. But note that the universe is maximally symmetric in space but not in all of the space-time so we should consider the space-time of the universe to be $\mathbf{R}\times\Sigma$, where $\mathbf{R}$ represents the time direction and Σ represents a maximally symmetric three-manifold.

5.2 Maximally Symmetric 3-D Space Metric

For a maximally symmetric space if we know the curvature of one point we are able to know the curvature everywhere because the curvature is the same everywhere in every direction. Since the geometry looks the same in every direction the curvature tensor should also look the same in all directions. For one point P on the maximally symmetric manifold in a locally inertial coordinate, the metric $\hat{g}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. Since the locally inertial coordinate is not unique we can perform a Lorentz transformation at point P and the metric remains to be $\eta_{\hat{\mu}\hat{\nu}}$. On the other hand, in a maximally symmetric space the Riemann curvature tensor $\hat{R}_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$ at point p is also unchanged under Lorentz transformations because the geometry looks the same in all directions. Therefore it is reasonable to guess that the Riemann tensor of a maximally symmetric manifold at point p somehow relates to the metric in these locally inertial coordinates. Actually, there is only one possibility

$$\hat{R}_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}(p) = \mathcal{K}(\hat{g}_{\hat{\rho}\hat{\mu}}(p)\hat{g}_{\hat{\sigma}\hat{\nu}}(p) - \hat{g}_{\hat{\rho}\hat{\nu}}(p)\hat{g}_{\hat{\sigma}\hat{\mu}}(p)) \tag{57}$$

considering the symmetry properties of the Riemann tensor. This is a tensorial equation so this must be true in any coordinate and since the manifold is maximally symmetric the Riemann tensor should be the same at any point. So we have

$$R_{\rho\sigma\mu\nu} = \mathcal{K}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{58}$$

Contract both sides once,

$$g^{\rho\mu}R_{\rho\sigma\mu\nu} = \mathcal{K}(g^{\rho\mu}g_{\rho\mu}g_{\sigma\nu} - g^{\rho\mu}g_{\rho\nu}g_{\sigma\mu}). \tag{59}$$

The left hand side is the Ricci tensor and the right hand side is $\mathcal{K}(ng_{\sigma\nu} - g_{\sigma\nu})$,

$$R_{\sigma\nu} = \mathcal{K}(ng_{\sigma\nu} - g_{\sigma\nu}). \tag{60}$$

Contract both sides twice,

$$g^{\sigma\nu}g^{\rho\mu}R_{\rho\sigma\mu\nu} = \mathcal{K}(g^{\sigma\nu}g^{\rho\mu}g_{\rho\mu}g_{\sigma\nu} - g^{\sigma\nu}g^{\rho\mu}g_{\rho\nu}g_{\sigma\mu}). \tag{61}$$

The left hand side is simply the Ricci scalar which is a constant over the manifold and the right hand side is $\mathcal{K}(n^2 - n)$,

$$R = \mathcal{K}(n^2 - n). \tag{62}$$

Therefore constant $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{R}{n(n-1)} \tag{63}$$

Hence, the geometry of a maximally symmetric manifold is specified by a constant curvature scalar R.

Our interest is in maximally symmetric three-dimensional space since we consider the space-time of the universe to have four dimensions with the time direction and the maximally symmetric three-dimensional space. As discussed before the maximally symmetric metric of the three-dimensional space should obey

$$^{(3)}R_{ijkl} = \mathcal{K}\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right), \tag{64}$$

and the Ricci scalar is

$$^{(3)}R_{jl} = 2\mathcal{K}g_{jl} \tag{65}$$

where γ_{ij} is the three-dimensional metric and $\mathcal{K}$ is given by

$$\mathcal{K} = \frac{R}{6}, \tag{66}$$

for $n = 3$.

A maximally symmetric space is guaranteed to be spherical symmetric and for spherical symmetric three-dimensional space the metric can be put into the form

$$dl^2 = \gamma_{ij}du^i du^j = e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2, \tag{67}$$

so this should also be true in the case of the maximally symmetric space. By calculating the Ricci tensor we find

$$^{(3)}R_{rr} = \frac{2}{r}\partial_r\beta = 2\mathcal{K}e^{2\beta(r)} \tag{68}$$

$$^{(3)}R_{\theta\theta} = e^{-2\beta}\left(r\partial_r\beta - 1\right) + 1 = 2\mathcal{K}r^2 \tag{69}$$

$$^{(3)}R_{\phi\phi} = \left[e^{-2\beta}\left(r\partial_r\beta - 1\right) + 1\right]\sin^2\theta = 2\mathcal{K}r^2 sin^2\theta, \tag{70}$$

and can solve for $\beta(r)$

$$\beta = -\frac{1}{2}\ln(1 - \mathcal{K}r^2), \tag{71}$$

Which gives the metric

$$dl^2 = \frac{dr^2}{1 - \mathcal{K}r^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{72}$$

$\mathcal{K}$ is a constant sets the curvature which can be positive, zero or negative. It is common to normalize the non-zero values. Do the substitution

$$r \to \frac{r}{\sqrt{|\mathcal{K}|}}, \tag{73}$$

and set

$$K|\mathcal{K}| = \mathcal{K}. \tag{74}$$

Thus $K \in \{+1, 0, -1\}$ and

$$dl^2 = \frac{1}{|\mathcal{K}|}\left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{75}$$

Where $\frac{1}{|\mathcal{K}|}$ is the curvature scalar which sets the physical size of the space. $K = +1$ corresponds to the closed space, $K = 0$ corresponds to the flat space and $K = -1$ corresponds to the open space.

Later we will see in the Robertson-Walker (RW) metric we can absorb the physical size into a scale factor which is a function of time so for convenience we conclude the metric of a maximally symmetric space as[1]

$$dl^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \quad K \in \{+1, 0, -1\}. \tag{76}$$

5.3 Robertson-Walker Metric

According to the cosmological principle the universe is homogeneous and isotropic on sufficiently large scales. In other words, the position of the Milky Way in the universe is statistically equivalent to the position of any other galaxies, and the universe looks statistically the same in all directions. Despite the cosmological principle being an approximation of our real universe, we consider in this section a cosmological model where the principle is exact. This cosmology is maximally symmetric in space but not in time, and is best described by the Robertson-Walker metric evolving according to Einstein's equation of General Relativity.

The metric of a spatially maximally symmetric universe expanding over time is given by the Robertson-Walker (RW) metric:

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] , \tag{77}$$

where the time coordinate t is the cosmic time, and the spherical coordinates r, θ and ϕ are called comoving coordinates. Any spatial slices of fixed t are maximally symmetric possessing six Killing vectors: three for spatial translations and three for spatial rotations. The cosmic time is hence interpreted as the proper time of a comoving observer who sees the universe spatially homogeneous and isotropic. The scale factor $R(t)$ is the only degree of freedom of the metric and it fully specifies the dynamics of the universe since it tells us how the distance between two points scales with time. The constant K is the spatial curvature and it describes the topology of the maximally symmetric hypersurface: a negative K describes a hyperbolic space, $K = 0$ describes a Euclidean space and a positive K describes a spherical space. The spherical space is compact and is therefore the only topology that admits a maximal radius. A universe with positive spatial curvature is then called "close" as opposed to "open" and "flat" for the case with negative and positive curvature respectively.

Considering the transformations

$$r \to \frac{r}{R(t_0)} , \qquad K \to KR^2(t_0) , \tag{78}$$

the RW metric keeps the same form as in Eq. (77) and it allows for a redefinition of the dimensionless scale factor a as

$$a(t) := \frac{R(t)}{R(t_0)}, \tag{79}$$

under which the RW assumes the standard form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] . \tag{80}$$

This metric and its physical implications as a homogeneous and isotropic expanding universe were first studied by Friedmann in the early 1920s.[23][24] Later, in the mid 1930s[25][26][27] Robertson and Walker derived it on the basis of isotropy and homogeneity. Lemaitre's work [28] had been also essential to develop the RW metric. It turns out that RW metric well describes our universe, in particular, on cosmological scales our universe looks homogeneous and isotropic as indicated by the analysis of the Sloan Digital Sky Survey (SDSS)[29] which concluded that the irregularities in the galaxy density are on the level of a few percent on scales of 100 Mpc.[30] Moreover, a recent work by Sarkar and Pandey [31] showed that the distribution of quasars is homogeneous on scales larger than $250h^{-1}Mpc$. Despite the deviations from homogeneity and isotropy being small on such large scales, a remarkable amount of information lies on such perturbations, and they will be the subject of the next chapter.[32][33]

5.4 The Friedmann Equations

From the property of maximal symmetry of the spatial slices of fixed cosmic time, it follows that any cosmic fields defined on the RW space-time are form-invariant under spatial rotations and translations [32]. Applying this remark to the energy-momentum tensor $T_{\mu\nu}$, which describes the matter content of the universe, implies that the only possibility allowed by symmetry is

$$T^{\mu}_{\ \nu}(t) = diag(-\rho(t), p(t), p(t), p(t)) \,, \tag{81}$$

where ρ can be interpreted as the energy density of a perfect fluid and p as the isotropic pressure. From the conservation of the energy-momentum tensor $\nabla_\mu T^{\mu}_{\ \nu} = 0$ we find

$$\dot{\rho} = -3H(p+\rho) \,, \tag{82}$$

where the dot stands for derivative with respect to time and H is the Hubble rate $H := \dot{a}/a$. The relation between gravity and such energy content is governed by Einstein's equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$. Given the RW metric, Friedmann equations can be straightforwardly computed from the Einstein equations. In the case under consideration of the RW metric Einstein's equations become[32]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \,, \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3} \,, \tag{83}$$

These equations are known as the Friedmann equations and are ordinary differential equations that can be solved for the scale factor $a(t)$ once the time evolution of the energy density and pressures is given.

In the history of the universe, there have been long periods of time where the total energy density was dominated by one type of component, e.g., radiation or matter. These periods of time are therefore called radiation dominated era (RDE) and matter dominated era (MDE) depending on which of these components was dominating.[22] In these cases it is straightforward to derive the dynamics of the scale factor from the Friedmann equations. Assuming an equation of state $p = w\rho$ for the perfect fluid, we can solve the conservation equation (82) to derive the scaling of the energy density

$$\rho(t) \propto a(t)^{-3(1+w)} \,, \tag{84}$$

which scales as a^{-4} for radiation ($w = 1/3$) and as a^{-3} for matter ($w = 0$). From this scaling, we immediately conclude that RDE takes place before MDE and the transition happens at matter-radiation equality. Plugging the solution above for the energy density in the second Friedmann equation in Eq. (83) we obtain the differential equation

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)} \,, \tag{85}$$

which can be integrated to derive

$$a \propto t^{\frac{2}{3(1+w)}} \,. \tag{86}$$

Therefore, we find that in RDE the scale factor evolves as $a \propto \sqrt{t}$ and in MDE as $a \propto t^{2/3}$. In the ΛCDM model, dark energy in the form of a cosmological constant is found to be dominating the energy budget of the universe from red-shift $z \approx 0.5$. Repeating the steps above for the equation of state $w = -1$ for a cosmological constant we obtain the scaling $a \propto e^{Ht}$ with constant Hubble rate. As a concluding remark, we note that the dynamics in RDE and MDE describe an expanding and decelerating universe, while in a dark energy dominated era the universe accelerates.[1]

6 Perturbations of the homogeneous universe

Our universe is homogeneous and isotropic on very large cosmological scales, where it can be described by RW metric. However, the cosmological principle is merely an approximation, which ignores all the structures present in the universe such as stars, galaxies, and also ourselves. To describe the real universe we can treat these inhomogeneities on cosmological scales as small deviations from the perfectly homogeneous universe. In other words, we can use a linearly perturbed Robertson-Walker metric with perturbations being of order 10^{-5} as measured in the CMB temperature anisotropies.[34] Moreover, given the measurement of the Planck data[35] for the curvature density parameter today $\Omega_{K,0} = 0.0008^{+0.0040}_{-0.0039}$, we can safely assume a spatially flat RW metric plus perturbations.

In this Chapter, we introduce the concept of perturbed RW cosmology and we perform some of the computations that will prove useful in the following Chapters.

6.1 Robertson-Walker Metric with Perturbations

For a spatially flat universe with $K = 0$ the RW metric in Eq. (80) takes the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,, \tag{87}$$

where we picked Cartesian coordinates. Instead of working with cosmic time t we introduce a new time coordinate, the conformal time η, defined by the following relation

$$\eta(t) = \int \frac{dt}{a(t)} \quad \rightarrow \quad dt = a(\eta)d\eta. \tag{88}$$

such that the flat RW metric becomes

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)\,, \tag{89}$$

which is nothing but a conformal Minkowski metric. To describe an inhomogeneous universe with metric $g_{\mu\nu}$ we consider the RW metric above as a fictitious background $\bar{g}_{\mu\nu}$, and we define the perturbations $\delta g_{\mu\nu} := g_{\mu\nu} - \bar{g}_{\mu\nu}$. Our convention for the perturbations in a generic metric is given here

$$g_{00} = -a^2(1 + 2A), \qquad g_{0a} = -a^2 B_a, \qquad g_{ab} = a^2\left(\delta_{ab} + 2C_{ab}\right)\,, \tag{90}$$

with A, B_a, and C_{ab} small perturbations that depend on RW space-time coordinates that can be decomposed in scalar-vector-tensor under spatial rotations:

$$A = \alpha \qquad B_a = \beta_{,a} + B^{(v)}_a\,, \qquad C_{ab} = \varphi\delta_{ab} + \gamma_{,a,b} + 2C^{(v)}_{(a,b)} + C^{(t)}_{ab} \tag{91}$$

The definition in Eq.(90) is valid at all orders in perturbations, but our computations will be limited to first order. At first order in perturbations the inverse metric $g^{\mu\nu}$ is

$$g^{00} = \frac{1}{a^2}(-1 + 2A), \qquad g^{0a} = -\frac{1}{a^2}B^a, \qquad g^{ab} = \frac{1}{a^2}\left(\delta^{ab} - 2C^{ab}\right)\,. \tag{92}$$

In the following part of this Thesis we will be interested in computing geometric quantities related to this perturbed universe. The Christoffel symbols defined as

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}\right)\,, \tag{93}$$

where the comma indicates partial derivative with respect to space-time coordinates, in the perturbed RW universe become

$$\begin{aligned}
\Gamma^0_{00} &= \frac{a'}{a} + A' , \qquad \Gamma^0_{0a} = A_{,a} - \frac{a'}{a}B_a , \qquad \Gamma^a_{00} = A^{,a} - B^{a\prime} - \frac{a'}{a}B^a \\
\Gamma^0_{ab} &= \frac{a'}{a}\delta_{ab} - 2\frac{a'}{a}\delta_{ab}A + B_{(a,b)} + C'_{ab} + 2\frac{a'}{a}C_{ab} , \quad \Gamma^a_{0b} = \frac{a'}{a}\delta^a_b + \frac{1}{2}\left(B_b{}^{,a} - B^a{}_{,b}\right) + C^{a\prime}_b , \\
\Gamma^a_{bc} &= \frac{a'}{a}\delta_{bc}B^a + C^a_{b,c} + C^a_{c,b} - C_{bc}{}^{,a}
\end{aligned} \tag{94}$$

From the Christoffel symbols we can compute the Riemann tensor defined as

$$R^\mu_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\epsilon_{\nu\sigma}\Gamma^\mu_{\rho\epsilon} - \Gamma^\epsilon_{\nu\rho}\Gamma^\mu_{\sigma\epsilon} , \tag{95}$$

that at first order in perturbations is

$$\begin{aligned}
&R^0_{000} = 0 , \qquad R^a_{000} = 0 , \qquad R^0_{b00} = 0 , \qquad R^a_{b00} = 0 , \qquad R^0_{0ab} = 0 \\
&R^0_{0a0} = -R^0_{00a} = \mathcal{H}'B_a \\
&R^a_{0b0} = -R^a_{00b} = -\mathcal{H}'\delta^a_b + \mathcal{H}A'\delta^a_b + A^{,a}{}_b - \frac{1}{2}\left(B_b{}^{,a} + B^a{}_{,b}\right)' - \frac{1}{2}\mathcal{H}\left(B_b{}^{,a} + B^a{}_{,b}\right) - C^{a\prime\prime}_b - \mathcal{H}C^{a\prime}_b \\
&R^0_{a0b} = -R^0_{ab0} = \mathcal{H}'\delta_{ab} - \left[\mathcal{H}A' + 2\mathcal{H}'A\right]\delta_{ab} - A_{,ab} + B'_{(a,b)} + \mathcal{H}B_{(a,b)} + C''_{ab} + \mathcal{H}C'_{ab} + 2\mathcal{H}'C_{ab} \\
&R^0_{abc} = 2\mathcal{H}\delta_{a[b}A_{,c]} + \frac{1}{2}\left(B_{c,ab} - B_{b,ac}\right) - 2C'_{a[b,c]} \\
&R^a_{0bc} = R^a_{0bc} = 2\mathcal{H}\delta^a_{[b}A_{,c]} - B_{[b}{}^{,a}{}_{,c]} + B^a{}_{,[bc]} - 2\mathcal{H}^2\delta^a_{[b}B_{c]} - 2C^{a\prime}_{[b,c]} \\
&R^a_{b0c} = -R^a_{bc0} = \mathcal{H}\left(\delta_{bc}A^{,a} - \delta^a_c A_{,b}\right) + \mathcal{H}'\delta_{bc}B^a - \mathcal{H}^2\left(\delta_{bc}B^a - \delta^a_c B_b\right) \\
&\qquad\qquad - \frac{1}{2}\left(B_b{}^{,a} - B^a{}_{,b}\right)_{,c} + C^{a\prime}_{c,b} - C'_{bc}{}^{,\alpha} \\
&R^a_{bcd} = \mathcal{H}^2\left(\delta^a_c\delta_{bd} - \delta^a_d\delta_{bc}\right)(1 - 2A) \\
&\qquad + \frac{1}{2}\mathcal{H}\left[\delta_{bd}\left(B_c{}^{,a} + B^a{}_{,c}\right) - \delta_{bc}\left(B_d{}^{,a} + B^a{}_{,d}\right) + 2\delta^a_c B_{(b,d)} - 2\delta^a_d B_{(b,c)}\right] \\
&\qquad + \mathcal{H}\left[\delta_{bd}C^{a\prime}_c - \delta_{bc}C^{a\prime}_d + \delta^a_c C'_{bd} - \delta^a_d C'_{bc} + 2\mathcal{H}\left(\delta^a_c C_{bd} - \delta^a_d C_{bc}\right)\right] \\
&\qquad + 2C^a_{(b,d),c} - 2C^a_{(b,c),d} + C_{bc}{}^{,a}{}_{,d} - C_{bd}{}^{,\alpha}{}_{,c}
\end{aligned} \tag{96}$$

The expressions for the linear-order Christoffel symbols and Riemann tensor above are computed in a generic gauge. However, depending on the problem under consideration it could be more convenient to work in a specific gauge. In cosmological perturbation theory, a gauge transformation is induced by a change of coordinates in the inhomogeneous universe

$$x^\mu \to \tilde{x}^\mu(x) = x^\mu + \xi^\mu(x) , \tag{97}$$

where ξ^μ is the generator of the small gauge transformation and it can be treated as a perturbation. The coordinate transformation induces a tensorial transformation of the metric components $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x})$ such that

$$g_{\alpha\beta}(x) = \frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}\tilde{g}_{\mu\nu}(\tilde{x}) , \tag{98}$$

with $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(\tilde{x})$ describing the same physics due to the diffeomorphism symmetry of Einstein's theory of General Relativity. At first order in perturbations, we derive the gauge transformation of the metric components[1]

$$\widetilde{\delta g}_{\mu\nu}(x) = \delta g_{\mu\nu}(x) - 2\xi_{(\mu;\nu)}(x) , \tag{99}$$

where the semicolon denotes the covariant derivative. Based on this equation, given a choice of the gauge field ξ^μ, we can set up to four functions in the metric perturbations to zero. One of the most popular gauges is the Newtonian gauge, obtained by setting the perturbations β and γ to zero

$$ds^2 = -a^2(1+2\alpha)d\eta^2 + a^2(1+2\varphi)(dx^2+dy^2+dz^2)\,, \tag{100}$$

where the potentials α and φ are related to the Newtonian potentials when taking the Newtonian limit of General Relativity.

The general covariance of general relativity guarantees that any coordinate system can be used to describe the physics and it has to be independent of coordinate systems. This is known as the diffeomorphism symmetry in general relativity. However, when we split the metric into the background and the perturbations around it by choosing a coordinate system, we explicitly change the correspondence of the physical Universe to the background homogeneous and isotropic Universe. Hence, the metric perturbations transform non-trivially (or gauge transform), and the diffeomorphism invariance implies that the physics should be gauge-invariant. We conclude by saying that there exist other ways to exploit the diffeomorphism symmetry of General Relativity by performing coordinate transformations that are not gauge transformations. These coordinate transformations do not assume that a splitting of the metric in background plus perturbations has been defined, and are hence valid independently of the chosen gauge. The next sections are devoted to such coordinate transformations and are the main focus of this thesis.[10]

Part IV
Fermi-Normal Coordinates

In Part II we have mentioned that it is always possible to find coordinates locally in which the metric looks like the metric of the Minkowski space-time to the first order. These coordinates are called locally inertial coordinates but we have not constructed any of these coordinates.

In 1922 Fermi showed that, given any curve in a Riemannian manifold, it is possible to introduce coordinates near this curve in such a way that the Christoffel symbols vanish along the curve, leaving the metric there rectangular.[36] If we specialize to the case where the curve is a geodesic, and choose a particular set of coordinates that satisfies the condition that along the curve the Christoffel symbols vanish, the resulting coordinates are called Fermi-Normal Coordinates because of an analogy to the Riemann normal coordinates.[37]

7 Basic Formalism and Analytical Formula

Let us first construct the Fermi-Normal coordinates. Consider a free-falling observer, whose trajectory is a time-like geodesic $x^\mu(\tau)$ parameterized by its proper time, which we call the central geodesic and it defines the spatial origin of the coordinate system at all times. We take the proper time τ along the central geodesic as the time coordinate. Then choose four orthonormal vectors (or tetrads) $[e_t]^\mu(\tau)$ and $[e_i]^\mu(\tau)$ on a point P on the geodesic. The four-velocity of the free-falling observer u^μ defines the time direction

$$[e_t]^\mu = u^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{101}$$

We then choose an orthonormal set of spatial basis vectors $[e_i]^\mu$ to fix the coordinate axes there such that the metric on point P on the geodesic is in the normal form $\eta_{\mu\nu}$. The orthonormal condition for the tetrads is

$$-1 = [e_t]^\mu [e_t]^\nu g_{\mu\nu}, \quad \delta_{ij} = [e_i]^\mu [e_j]^\nu g_{\mu\nu} \tag{102}$$

Since $[e_t]^\mu$ is the tangent vector of the geodesic it is parallel transported along the geodesic by the definition of the geodesic. The three space-like tetrads $[e_i]^\mu$ are also defined all along the geodesic by parallel transporting them. Which implies

$$[e_i]^\mu{}_{;\tau} = 0 \tag{103}$$

Since parallel transport preserves angles the metric in the coordinate fixed by the four tetrads is $\eta_{\mu\nu}$ all along the geodesic.

At the point P along the time like geodesic, consider another point Q with definite proper distance s_Q (fixed value) that is uniquely connected by another space like geodesic $x^\mu(s)$ from $x^\mu(s=0) = P$. The spatial tangent vectors of the space like geodesic at point P can be written as the linear combination of the spatial basis vectors

$$\left.\frac{dx^\mu}{ds}\right|_{s=0} = a^i [e_i]^\mu \tag{104}$$

Now do the Taylor's expansion of $x^\mu(s_Q)$ for small s_Q around $s = 0$, $x^\mu(s_Q)$ can be expanded as

$$x^\mu(s_Q) = x^\mu(0) + s_Q \left.\frac{dx^\mu(s)}{ds}\right|_0 + \frac{1}{2} s_Q^2 \left.\frac{d^2x^\mu(s)}{ds^2}\right|_0 + \frac{1}{6} s_Q^3 \left.\frac{d^3x^\mu(s)}{ds^3}\right|_0 \tag{105}$$

Further define

$$x_F^i = a^i s_Q, \tag{106}$$

so that $x^\mu(s_Q)$ can be written as

$$\begin{aligned} x_Q^\mu =& P + [e_i]_P^\mu x_F^i - \frac{1}{2} \Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_j]_P^\beta x_F^i x_F^j \\ & - \frac{1}{6} \left[\Gamma^\mu_{\alpha\beta,\gamma} - 2\Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P [e_i]_P^\alpha [e_j]_P^\beta [e_k]_P^\gamma x_F^i x_F^j x_F^k \end{aligned} \tag{107}$$

Where we have use the geodesic equation.

To compute the metric in the FNC, we need to compute the derivatives of the coordinate transformation:

$$\begin{aligned} \frac{\partial x^\mu}{\partial x_F^0}(Q) &= \frac{\partial x^\mu}{\partial x_F^0}(P) + \frac{\partial}{\partial x_F^0}\left([e_i]_P^\mu\right) x_F^i - \frac{1}{2}\frac{\partial}{\partial x_F^0}\left[\Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_j]_P^\beta\right] x_F^i x_F^j + \mathcal{O}\left(x_F^3\right) \\ &= [e_t]_P^\mu - \Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_t]_P^\beta x_F^i - \frac{1}{2}\left[\Gamma^\mu_{\alpha\beta,\gamma} - 2\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P [e_t]_P^\gamma [e_i]_P^\alpha [e_j]_P^\beta x_F^i x_F^j \end{aligned} \tag{108}$$

The spatial derivatives are:

$$
\begin{aligned}
\frac{\partial x^\mu}{\partial x_F^l}(Q) &= \frac{\partial x^\mu}{\partial x_F^l}(P) + \frac{\partial}{\partial x_F^l}\left([e_i]_P^\mu x_F^i\right) - \frac{1}{2}\frac{\partial}{\partial x_F^l}\left(\Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_j]_P^\beta x_F^i x_F^j\right) \\
&\quad - \frac{1}{6}\frac{\partial}{\partial x_F^l}\left(\left[\Gamma^\mu_{\alpha\beta,\gamma} - 2\Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P [e_i]_P^\alpha [e_j]_P^\beta [e_k]_P^\gamma x_F^i x_F^j x_F^k\right) \\
&= 0 + [e_l]_P^\mu - \Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_l]_P^\beta x_F^i - \frac{1}{6}\left[\Gamma^\mu_{\alpha\beta,\gamma} - 2\Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P \\
&\quad \left[[e_l]_P^\alpha [e_j]_P^\beta [e_k]_P^\gamma x_F^j x_F^k + [e_i]_P^\alpha [e_l]_P^\beta [e_k]_P^\gamma x_F^i x_F^k + [e_i]_P^\alpha [e_j]_P^\beta [e_l]_P^\gamma x_F^i x_F^j\right] \\
&= [e_l]_P^\mu - \Gamma^\mu_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_l]_P^\beta x_F^i \\
&\quad - \frac{1}{6}\left[\Gamma^\mu_{\alpha\beta,\gamma} + 2\Gamma^\mu_{\gamma\alpha,\beta} - 2\Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta} - 4\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P [e_i]_P^\alpha [e_j]_P^\beta [e_l]_P^\gamma x_F^i x_F^j
\end{aligned} \tag{109}
$$

Expand global metric at Q around P :

$$
\begin{aligned}
g_{\alpha\beta}(Q) &= g_{\alpha\beta}|_P + g_{\alpha\beta,\mu}|_P [x_Q^\mu - x_P^\mu] + \frac{1}{2} g_{\alpha\beta,\mu\nu}\Big|_P [x_Q^\mu - x_P^\mu][x_Q^\nu - x_P^\nu] \\
&= g_{\alpha\beta}|_P + g_{\alpha\beta,\mu}|_P [e_i]_P^\mu x_F^i + \frac{1}{2}\left[g_{\alpha\beta,\mu\nu} - g_{\alpha\beta,\sigma}\Gamma^\sigma_{\mu\nu}\right]_P [e_i]_P^\mu [e_j]_P^\nu x_F^i x_F^j
\end{aligned} \tag{110}
$$

The FNC metric at Q can be calculated now with $g^F_{\mu\nu}(Q) = \frac{\partial x^\alpha}{\partial x_F^\mu}\frac{\partial x^\beta}{\partial x_F^\nu} g_{\alpha\beta}(Q)$.

$$
\begin{aligned}
g^F_{00}(Q) &= [e_t]_P^\alpha [e_t]_P^\beta g_{\alpha\beta} + \left(g_{\alpha\beta,\rho} - g_{\sigma\beta}\Gamma^\sigma_{\rho\alpha} - g_{\alpha\sigma}\Gamma^\sigma_{\rho\beta}\right)_P [e_t]_P^\alpha [e_t]_P^\beta [e_i]_P^\rho x_F^i \\
&\quad + \left(\frac{1}{2} g_{\mu\nu,\alpha\beta} - \frac{1}{2} g_{\mu\nu,\sigma}\Gamma^\sigma_{\alpha\beta} - 2 g_{\mu\gamma,\alpha}\Gamma^\gamma_{\beta\nu} - g_{\mu\gamma}\Gamma^\gamma_{\alpha\beta,\nu} + 2 g_{\mu\gamma}\Gamma^\gamma_{\sigma\beta}\Gamma^\sigma_{\nu\alpha} + g_{\gamma\sigma}\Gamma^\gamma_{\alpha\mu}\Gamma^\sigma_{\beta\nu}\right)_P \\
&\quad [e_t]_P^\mu [e_t]_P^\nu [e_l]_P^\alpha [e_m]_P^\beta x_F^l x_F^m \\
&= \eta_{00} - R^F_{0l0m} x_F^l x_F^m
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
g^F_{0a}(Q) &= [e_t]_P^\alpha [e_a]_P^\beta g_{\alpha\beta} + \left(g_{\alpha\beta,\rho} - g_{\sigma\beta}\Gamma^\sigma_{\rho\alpha} - g_{\alpha\sigma}\Gamma^\sigma_{\rho\beta}\right)_P [e_t]_P^\alpha [e_a]_P^\beta [e_i]_P^\rho x_F^i \\
&\quad + \left[\frac{1}{2} g_{\mu\nu,\alpha\beta} - \frac{1}{2} g_{\mu\nu,\sigma}\Gamma^\sigma_{\alpha\beta} - g_{\mu\gamma,\alpha}\Gamma^\gamma_{\beta\nu} - g_{\nu\gamma,\alpha}\Gamma^\gamma_{\beta\mu} - \frac{1}{2} g_{\nu\gamma}\Gamma^\gamma_{\alpha\beta,\mu} + g_{\nu\gamma}\Gamma^\gamma_{\sigma\alpha}\Gamma^\sigma_{\mu\beta} + g_{\gamma\sigma}\Gamma^\gamma_{\alpha\mu}\Gamma^\sigma_{\beta\nu}\right. \\
&\quad \left. - \frac{1}{6} g_{\mu\lambda}\left(\Gamma^\lambda_{\alpha\beta,\nu} + 2\Gamma^\lambda_{\nu\alpha,\beta} - 2\Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\alpha\beta} - 4\Gamma^\lambda_{\sigma\beta}\Gamma^\sigma_{\alpha\nu}\right)\right]_P [e_t]_P^\mu [e_a]_P^\nu [e_l]_P^\alpha [e_m]_P^\beta x_F^l x_F^m \\
&= \eta_{0a} - \frac{2}{3} R^F_{0lam} x_F^l x_F^m
\end{aligned} \tag{112}
$$

$$
\begin{aligned}
g^F_{ab}(Q) &= [e_a]_P^\alpha [e_b]_P^\beta g_{\alpha\beta} + \left(g_{\alpha\beta,\rho} - g_{\sigma\beta}\Gamma^\sigma_{\rho\alpha} - g_{\alpha\sigma}\Gamma^\sigma_{\rho\beta}\right)_P [e_a]_P^\alpha [e_b]_P^\beta [e_i]_P^\rho x_F^i \\
&\quad + \left[\frac{1}{2} g_{\mu v,\alpha\beta} - \frac{1}{2} g_{\mu v,\sigma}\Gamma^\sigma_{\alpha\beta} - 2 g_{\mu\gamma,\alpha}\Gamma^\gamma_{v\beta} + g_{\gamma\sigma}\Gamma^\gamma_{\alpha\mu}\Gamma^\sigma_{\beta v}\right. \\
&\quad \left. - \frac{1}{3} g_{\mu\lambda}\left(\Gamma^\lambda_{\alpha\beta,v} + 2\Gamma^\lambda_{v\alpha,\beta} - 2\Gamma^\lambda_{\sigma v}\Gamma^\sigma_{\alpha\beta} - 4\Gamma^\lambda_{\sigma\beta}\Gamma^\sigma_{\alpha v}\right)\right]_P [e_a]_P^\mu [e_b]_P^v [e_l]_P^\alpha [e_m]_P^\beta x_F^l x_F^m \\
&= \eta_{ab} - \frac{1}{3} R^F_{albm} x_F^l x_F^m
\end{aligned} \tag{113}
$$

Therefore, we conclude that

$$
g^F_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(R^F_{\mu l\nu m} x_F^l x_F^m). \tag{114}
$$

Where

$$
R^F_{\alpha\beta\gamma\delta} = [e_\alpha]_P^\mu [e_\beta]_P^\nu [e_\gamma]_P^\kappa [e_\delta]_P^\lambda R_{\mu\nu\kappa\lambda} \tag{115}
$$

is the Riemann tensor at point P in FNC.

In this way we have constructed a coordinate system in which the coordinate axes are fixed by the four tetrads defined along the time-like geodesic of a free-falling observer and the components of the coordinate for a point near the central geodesic are defined by the proper distance. FNC is the inertial coordinate that a free-falling observer would set in a neighborhood around them all along the geodesic. On the geodesic the metric $g^F_{\mu\nu}$ is exactly $\eta_{\mu\nu}$ and for the point in the neighborhood the metric is the Minkowski one plus some deviations.[38]

8 Linear Order Calculations

We have now constructed the Fermi normal coordinates on an arbitrary manifold with any global coordinate but the universe is described by the Robertson-Walker metric with perturbations. Therefore, we would like to use the perturbated Robertson-Walker metric as the global metric and then apply the Fermi normal coordinates.

First, we find the four tetrads in the Robertson-Walker metric. Since the universe is only homogeneous and isotropic to the comoving observer whose four-velocity is given by $u^\mu = \frac{1}{a}(1,0,0,0)$ and a comoving observer is a free-falling observer we would like to use the time-like geodesic of the comoving observer as the central geodesic. Therefore, the tetrad of the time direction is simply

$$[\bar{e}_t]^\mu = \frac{1}{a}(1,0,0,0). \tag{116}$$

The spatial directions can be arbitrary. For convenience, however, we fix the spatial symmetry by aligning the spatial tetrads directions with the coordinate directions used in the Robertson-Walker metric in a homogeneous and isotropic universe as

$$[\bar{e}_i]^\mu = \frac{1}{a}(0,\delta^a_i). \tag{117}$$

However, the real universe is not perfectly homogeneous and isotropic so the free-falling geodesic will have some small deviations from the one of the perfectly homogeneous and isotropic universe. Therefore, the tetrads will not align with the coordinate directions used in the Robertson-Walker metric anymore. We need to consider small perturbations on the tetrads, the tetrads with perturbations can be written as

$$[e_t]^\mu = \frac{1}{a}\left(1-A, V^a\right), \tag{118}$$

and

$$[e_i]^\mu = \frac{1}{a}\left(V_i - B_i, \delta^a_i - C^a_i - \epsilon^a{}_{ij}\Omega^j\right). \tag{119}$$

Where V^a is the spatial velocity and Ω^j is the rotation of spatial axis.[39]

8.1 The coordinates $x^\mu_Q(x^i_F)$ of the point Q

We have introduced the perturbations terms into the background metric and we have written down the four orthonormal tetrads $[e_\alpha]^\mu$ with perturbations. Now we are prepared to derive the expression for the global coordinate x^μ_Q at the point Q in terms of Fermi coordinates up to $\mathcal{O}\left(x^3_F\right)$.

We have calculated in section 7 that

$$x^\mu_Q = P + [e_i]^\mu_P\, x^i_F - \frac{1}{2}\Gamma^\mu_{\alpha\beta}\Big|_P [e_i]^\alpha_P\, [e_j]^\beta_P\, x^i_F x^j_F - \frac{1}{6}\left[\Gamma^\mu_{\alpha\beta,\gamma} - 2\Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P [e_i]^\alpha_P\, [e_j]^\beta_P\, [e_k]^\gamma_P\, x^i_F x^j_F x^k_F \tag{120}$$

Where the tetrads are those with perturbations defined before and the Christoffel symbols are based on the perturbed Roberston-Walker metric which has been calculated in section 6.

Now we will calculate all the components of the coordinates x^μ_Q. First we find the time component x^0_Q.

$$x^0_Q = x^0_P + [e_i]^0_P\, x^i_F - \frac{1}{2}\Gamma^0_{\alpha\beta}\Big|_P [e_i]^\alpha_P\, [e_j]^\beta_P\, x^i_F x^j_F - \frac{1}{6}\left[\Gamma^0_{\alpha\beta,\gamma} - 2\Gamma^0_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P [e_i]^\alpha_P\, [e_j]^\beta_P\, [e_k]^\gamma_P\, x^i_F x^j_F x^k_F\,, \tag{121}$$

we then expand the term quadratic in Fermi coordinates as follows

$$\begin{aligned}
&-\frac{1}{2}\Gamma^0_{\alpha\beta}\Big|_P [e_i]^\alpha_P\, [e_j]^\beta_P\, x^i_F x^j_F \\
&= -\frac{1}{2}\left\{\Gamma^0_{00}\,[e_i]^0\,[e_j]^0 + \Gamma^0_{a0}\,[e_i]^a\,[e_j]^0 + \Gamma^0_{0b}\,[e_i]^0\,[e_j]^b + \Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\right\} x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left(\frac{a'}{a}\delta_{ab} - 2\frac{a'}{a}\delta_{ab}A + B_{(a,b)} + C'_{ab} + 2\frac{a'}{a}C_{ab}\right)\left(\delta^a_i - C^a_i - \epsilon^a{}_{il}\Omega^l\right)\left(\delta^b_j - C^b_j - \epsilon^b{}_{jl}\Omega^l\right) x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[\frac{a'}{a}\delta_{ab}(\delta^a_i - C^a_i - \epsilon^a{}_{il}\Omega^l)\delta^b_j + \frac{a'}{a}\delta_{ab}\delta^a_i(-C^b_j - \epsilon^b{}_{jl}\Omega^l) + \right. \\
&\left.\left(-2\frac{a'}{a}\delta_{ab}A + B_{(a,b)} + C'_{ab} + 2\frac{a'}{a}C_{ab}\right)\delta^a_i\delta^b_j\right] x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[\frac{a'}{a}\left(\delta_{ij} - C_{ij} - \varepsilon_{jil}\Omega^l\right) + \frac{a'}{a}\left(-C_{ij} - \varepsilon_{ijl}\Omega^l\right) - 2\frac{a'}{a}\delta_{ij}A + B_{(i,j)} + C'_{ij} + 2\frac{a'}{a}C_{ij}\right] x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[\mathcal{H}\delta_{ij} - 2\mathcal{H}\delta_{ij}A + B_{(i,j)} + C'_{ij}\right] x^i_F x^j_F\,,
\end{aligned} \tag{122}$$

and for the two terms cubic in the Fermi coordinates we obtain

$$\begin{aligned}
&\Gamma^0_{\alpha\beta,\gamma}\,[e_i]^\alpha\,[e_j]^\beta\,[e_k]^\gamma\, x^i_F x^j_F x^k_F \\
&= \left\{\Gamma^0_{00,0}\,[e_i]^0\,[e_j]^0\,[e_k]^0 + \Gamma^0_{a0,0}\,[e_i]^a\,[e_j]^0\,[e_k]^0 + \Gamma^0_{0b,0}\,[e_i]^0\,[e_j]^b\,[e_k]^0 + \Gamma^0_{00,c}\,[e_i]^0\,[e_j]^0\,[e_k]^c \right.\\
&\quad\left. + \Gamma^0_{ab,0}\,[e_i]^a\,[e_j]^b\,[e_k]^0 + \Gamma^0_{a0,c}\,[e_i]^a\,[e_j]^0\,[e_k]^c + \Gamma^0_{0b,c}\,[e_i]^0\,[e_j]^b\,[e_k]^c + \Gamma^0_{ab,c}\,[e_i]^a\,[e_j]^b\,[e_k]^c\right\} \\
&\quad x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left[\mathcal{H}'\delta_{ab}\delta^a_i\delta^b_j(V_k - B_k) + (-2\mathcal{H}\delta_{ab}A_{,c} + B_{(a,b),c} + C'_{ab,c} + 2\mathcal{H}C_{ab,c})\delta^a_i\delta^b_j\delta^c_k\right] x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left[\mathcal{H}'\delta_{ij}(V_k - B_k) - 2\mathcal{H}\delta_{ij}A_{,k} + B_{(i,j),k} + C'_{ij,k} + 2\mathcal{H}C_{ij,k}\right] x^i_F x^j_F x^k_F
\end{aligned} \tag{123}$$

and

$$
\begin{aligned}
& -2\Gamma^0_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\left[e_i\right]^\alpha\left[e_j\right]^\beta\left[e_k\right]^\gamma x^i_F x^j_F x^k_F \\
&= -2\Big\{\Gamma^0_{\sigma 0}\Gamma^\sigma_{00}\left[e_i\right]^0\left[e_j\right]^0\left[e_k\right]^0 + \Gamma^0_{\sigma a}\Gamma^\sigma_{00}\left[e_i\right]^a\left[e_j\right]^0\left[e_k\right]^0 + \Gamma^0_{\sigma 0}\Gamma^\sigma_{b0}\left[e_i\right]^0\left[e_j\right]^b\left[e_k\right]^0 \\
&\quad + \Gamma^0_{\sigma 0}\Gamma^\sigma_{0c}\left[e_i\right]^0\left[e_j\right]^0\left[e_k\right]^c + \Gamma^0_{\sigma a}\Gamma^\sigma_{b0}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^0 + \Gamma^0_{\sigma a}\Gamma^\sigma_{0c}\left[e_i\right]^a\left[e_j\right]^0\left[e_k\right]^c + \Gamma^0_{\sigma 0}\Gamma^\sigma_{bc}\left[e_i\right]^0\left[e_j\right]^b\left[e_k\right]^c \\
&\quad + \Gamma^0_{\sigma a}\Gamma^\sigma_{bc}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^c\Big\} x^i_F x^j_F x^k_F \\
&= -2\Big\{\Gamma^0_{0a}\Gamma^0_{b0}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^0 + \Gamma^0_{da}\Gamma^d_{b0}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^0 + \Gamma^0_{0a}\Gamma^0_{0c}\left[e_i\right]^a\left[e_j\right]^0\left[e_k\right]^c \\
&\quad + \Gamma^0_{da}\Gamma^d_{0c}\left[e_i\right]^a\left[e_j\right]^0\left[e_k\right]^c + \Gamma^0_{00}\Gamma^0_{bc}\left[e_i\right]^0\left[e_j\right]^b\left[e_k\right]^c + \Gamma^0_{d0}\Gamma^d_{bc}\left[e_i\right]^0\left[e_j\right]^b\left[e_k\right]^c + \Gamma^0_{0a}\Gamma^0_{bc}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^c \\
&\quad + \Gamma^0_{da}\Gamma^d_{bc}\left[e_i\right]^a\left[e_j\right]^b\left[e_k\right]^c\Big\} x^i_F x^j_F x^k_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}\delta_{da}\mathcal{H}\delta^d_b\delta^a_i\delta^b_j(V_k-B_k) + \mathcal{H}\delta_{da}\mathcal{H}\delta^d_c\delta^a_i\delta^c_k(V_j-B_j) + \mathcal{H}\mathcal{H}\delta_{bc}(V_i-B_i)\delta^b_j\delta^c_k \\
&\qquad + (A_{,a}-\mathcal{H}B_a)\mathcal{H}\delta_{bc}\delta^a_i\delta^b_j\delta^c_k + \mathcal{H}\delta_{da}(\mathcal{H}\delta_{bc}B^d + C^d_{b,c} + C^d_{c,b} - C_{bc}{}^{,d})\delta^a_i\delta^b_j\delta^c_k\Big] x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left[-6\mathcal{H}^2\delta_{ij}(V_k-B_k) - 2\mathcal{H}\delta_{jk}A_{,i} + 2\mathcal{H}^2\delta_{jk}B_i - 2\mathcal{H}^2\delta_{jk}B_i - 2\mathcal{H}C_{ij,k} - 2\mathcal{H}C_{ik,j}\right. \\
&\qquad \left. + 2\mathcal{H}C_{jk,i}\right] x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left[-6\mathcal{H}^2\delta_{ij}(V_k-B_k) - 2\mathcal{H}\delta_{jk}A_{,i} - 2\mathcal{H}C_{ij,k}\right] x^i_F x^j_F x^k_F
\end{aligned}
\tag{124}
$$

Thus, the third order term is

$$
\begin{aligned}
& -\frac{1}{6}\left[\Gamma^0_{\alpha\beta,\gamma} - 2\Gamma^0_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]\left[e_i\right]^\alpha\left[e_j\right]^\beta\left[e_k\right]^\gamma x^i_F x^j_F x^k_F \\
&= -\frac{1}{6a^3}\left[\mathcal{H}'\delta_{ij}(V_k-B_k) - 2\mathcal{H}\delta_{ij}A_{,k} + B_{(i,j),k} + C'_{ij,k} + 2\mathcal{H}C_{ij,k} - 6\mathcal{H}^2\delta_{ij}(V_k-B_k)\right. \\
&\qquad \left. - 2\mathcal{H}\delta_{jk}A_{,i} - 2\mathcal{H}C_{ij,k}\right] x^i_F x^j_F x^k_F \\
&= -\frac{1}{6a^3}\left[\mathcal{H}'\delta_{ij}(V_k-B_k) - 6\mathcal{H}^2\delta_{ij}(V_k-B_k) - 4\mathcal{H}\delta_{ij}A_{,k} + B_{(i,j),k} + C'_{ij,k}\right] x^i_F x^j_F x^k_F
\end{aligned}
\tag{125}
$$

Finally we get the expansion of x^0_Q in terms of the Fermi normal coordinate as

$$
\begin{aligned}
x^0_Q =& x^0_P + \frac{1}{a}(V_i-B_i)x^i_F - \frac{1}{2a^2}\left[\mathcal{H}\delta_{ij} - 2\mathcal{H}\delta_{ij}A + B_{(i,j)} + C'_{ij}\right] x^i_F x^j_F \\
& - \frac{1}{6a^3}\left[\mathcal{H}'\delta_{ij}(V_k-B_k) - 6\mathcal{H}^2\delta_{ij}(V_k-B_k) - 4\mathcal{H}\delta_{ij}A_{,k} + B_{(i,j),k} + C'_{ij,k}\right] x^i_F x^j_F x^k_F
\end{aligned}
\tag{126}
$$

Then we find the spatial components x^l_Q

$$
\begin{aligned}
x^l_Q =& x^l_P + \left[e_i\right]^l_P x^i_F - \frac{1}{2}\Gamma^l_{\alpha\beta}\Big|_P \left[e_i\right]^\alpha_P\left[e_j\right]^\beta_P x^i_F x^j_F \\
& - \frac{1}{6}\left[\Gamma^l_{\alpha\beta,\gamma} - 2\Gamma^l_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P \left[e_i\right]^\alpha_P\left[e_j\right]^\beta_P\left[e_k\right]^\gamma_P x^i_F x^j_F x^k_F
\end{aligned}
\tag{127}
$$

The first order term is

$$
\left[e_i\right]^l_P x^i_F = \frac{1}{a}(\delta^l_i - C^l_i - \epsilon^l{}_{ih}\Omega^h)x^i_F
\tag{128}
$$

The second order term is

$$
\begin{aligned}
&-\frac{1}{2}\Gamma^l_{\alpha\beta}\bigg|_P [e_i]^\alpha_P [e_j]^\beta_P x^i_F x^j_F \\
&= -\frac{1}{2}\left\{\Gamma^l_{00}[e_i]^0[e_j]^0 + \Gamma^l_{a0}[e_i]^a[e_j]^0 + \Gamma^l_{0b}[e_i]^0[e_j]^b + \Gamma^l_{ab}[e_i]^a[e_j]^b\right\} x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[\mathcal{H}\delta^l_a\delta^a_i(V_j - B_j) + \mathcal{H}\delta^l_b\delta^b_j(V_i - B_i) + (\mathcal{H}\delta_{ab}B^l + C^l_{a,b} + C^l_{a,b} - C_{ab}{}^{,l})\delta^a_i\delta^b_j\right] x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[\mathcal{H}\delta^l_i(V_j - B_j) + \mathcal{H}\delta^l_j(V_i - B_i) + \mathcal{H}\delta_{ij}B^l + C^l_{i,j} + C^l_{i,j} - C_{ij}{}^{,l}\right] x^i_F x^j_F \\
&= -\frac{1}{2a^2}\left[2\mathcal{H}\delta^l_i(V_j - B_j) + \mathcal{H}\delta_{ij}B^l + C^l_{i,j} + C^l_{i,j} - C_{ij}{}^{,l}\right] x^i_F x^j_F
\end{aligned}
\tag{129}
$$

To compute the third order term by parts, the first term is

$$
\begin{aligned}
&\Gamma^l_{\alpha\beta,\gamma}[e_i]^\alpha[e_j]^\beta[e_k]^\gamma x^i_F x^j_F x^k_F \\
&= \Big\{\Gamma^l_{00,0}[e_i]^0[e_j]^0[e_k]^0 + \Gamma^l_{a0,0}[e_i]^a[e_j]^0[e_k]^0 + \Gamma^l_{0b,0}[e_i]^0[e_j]^b[e_k]^0 + \Gamma^l_{00,c}[e_i]^0[e_j]^0[e_k]^c \\
&\quad + \Gamma^l_{ab,0}[e_i]^a[e_j]^b[e_k]^0 + \Gamma^l_{a0,c}[e_i]^a[e_j]^0[e_k]^c + \Gamma^l_{0b,c}[e_i]^0[e_j]^b[e_k]^c + \Gamma^l_{ab,c}[e_i]^a[e_j]^b[e_k]^c\Big\} \\
&\quad x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left[\left(\mathcal{H}\delta_{ab}B^l_{,c} + C^l_{a,bc} + C^l_{b,ac} - C_{ab,c}{}^{,l}\right)\delta^a_i\delta^b_j\delta^c_k\right] x^i_F x^j_F x^k_F \\
&= \frac{1}{a^3}\left(\mathcal{H}\delta_{ij}B^l_{,k} + C^l_{i,jk} + C^l_{j,ik} - C_{ij,k}{}^{,l}\right) x^i_F x^j_F x^k_F
\end{aligned}
\tag{130}
$$

and the second term is

$$
\begin{aligned}
& -2\Gamma^l_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\,[e_i]^\alpha\,[e_j]^\beta\,[e_k]^\gamma\,x^i_Fx^j_Fx^k_F \\
&= -2\Big\{\Gamma^l_{\sigma0}\Gamma^\sigma_{00}\,[e_i]^0\,[e_j]^0\,[e_k]^0 + \Gamma^l_{\sigma a}\Gamma^\sigma_{00}\,[e_i]^a\,[e_j]^0\,[e_k]^0 + \Gamma^l_{\sigma0}\Gamma^\sigma_{b0}\,[e_i]^0\,[e_j]^b\,[e_k]^0 \\
&\quad + \Gamma^l_{\sigma0}\Gamma^\sigma_{0c}\,[e_i]^0\,[e_j]^0\,[e_k]^c + \Gamma^l_{\sigma a}\Gamma^\sigma_{b0}\,[e_i]^a\,[e_j]^b\,[e_k]^0 + \Gamma^l_{\sigma a}\Gamma^\sigma_{0c}\,[e_i]^a\,[e_j]^0\,[e_k]^c + \Gamma^l_{\sigma0}\Gamma^\sigma_{bc}\,[e_i]^0\,[e_j]^b\,[e_k]^c \\
&\quad + \Gamma^l_{\sigma a}\Gamma^\sigma_{bc}\,[e_i]^a\,[e_j]^b\,[e_k]^c\Big\}\,x^i_Fx^j_Fx^k_F \\
&= -2\Big\{\Gamma^l_{0a}\Gamma^0_{b0}\,[e_i]^a\,[e_j]^b\,[e_k]^0 + \Gamma^l_{da}\Gamma^d_{b0}\,[e_i]^a\,[e_j]^b\,[e_k]^0 + \Gamma^l_{0a}\Gamma^0_{0c}\,[e_i]^a\,[e_j]^0\,[e_k]^c \\
&\quad + \Gamma^l_{da}\Gamma^d_{0c}\,[e_i]^a\,[e_j]^0\,[e_k]^c + \Gamma^l_{00}\Gamma^0_{bc}\,[e_i]^0\,[e_j]^b\,[e_k]^c + \Gamma^l_{d0}\Gamma^d_{bc}\,[e_i]^0\,[e_j]^b\,[e_k]^c + \Gamma^l_{0a}\Gamma^0_{bc}\,[e_i]^a\,[e_j]^b\,[e_k]^c \\
&\quad + \Gamma^l_{da}\Gamma^d_{bc}\,[e_i]^a\,[e_j]^b\,[e_k]^c\Big\}\,x^i_Fx^j_Fx^k_F \\
&= -2\frac{1}{a^3}\Big[(\mathcal{H}\delta^l_a + \frac{1}{2}\left({B_a}^{,l} - {B^l}_{,a}\right) + C^{l\prime}_a)(\mathcal{H}\delta_{bc} - 2\mathcal{H}\delta_{bc}A + B_{(b,c)} + C'_{bc} + 2\mathcal{H}C_{bc}) \\
&\qquad (\delta^a_i - C^a_i - {\epsilon^a}_{ih}\Omega^h)(\delta^b_j - C^b_j - {\epsilon^b}_{jh}\Omega^h)(\delta^c_k - C^c_k - {\epsilon^c}_{kh}\Omega^h)\Big]\,x^i_Fx^j_Fx^k_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}^2\delta^l_a\delta_{bc}\delta^a_i\delta^b_j\delta^c_k + (\frac{1}{2}\left({B_a}^{,l} - {B^l}_{,a}\right) + C^{l\prime}_a)\mathcal{H}\delta_{bc}\delta^a_i\delta^b_j\delta^c_k \\
&\qquad + \mathcal{H}\delta^l_a(-2\mathcal{H}\delta_{bc}A + B_{(b,c)} + C'_{bc} + 2\mathcal{H}C_{bc})\delta^a_i\delta^b_j\delta^c_k + \mathcal{H}^2\delta^l_a\delta_{bc}(-C^a_i - {\epsilon^a}_{ih}\Omega^h)\delta^b_j\delta^c_k \\
&\qquad + \mathcal{H}^2\delta^l_a\delta_{bc}\delta^a_i\delta^c_k(-C^b_j - {\epsilon^b}_{jh}\Omega^h) + \mathcal{H}^2\delta^l_a\delta_{bc}\delta^a_i\delta^b_j(-C^c_k - {\epsilon^c}_{kh}\Omega^h)\Big]\,x^i_Fx^j_Fx^k_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}^2\delta^l_i\delta_{jk} + \mathcal{H}\delta_{jk}(\frac{1}{2}{B_i}^{,l} - \frac{1}{2}{B^l}_{,i} + C^{l\prime}_i) - 2\mathcal{H}^2\delta^l_i\delta_{jk}A + \mathcal{H}\delta^l_i(B_{(j,k)} + C'_{jk} + 2\mathcal{H}C_{jk}) \\
&\qquad + \mathcal{H}^2\delta_{jk}(-C^l_i - {\epsilon^l}_{ih}\Omega^h) + \mathcal{H}^2\delta^l_i(-C_{jk} - \epsilon_{kjh}\Omega^h) + \mathcal{H}^2\delta^l_i(-C_{jk} - \epsilon_{jkh}\Omega^h)\Big]\,x^i_Fx^j_Fx^k_F \\
&= \frac{1}{a^3}\Big[-2\mathcal{H}^2\delta^l_i\delta_{jk}(1-2A) - \mathcal{H}\delta_{jk}({B_i}^{,l} - {B^l}_{,i} + 2C^{l\prime}_i) - 4\mathcal{H}^2\delta^l_iC_{jk} - \mathcal{H}\delta^l_i(2B_{(j,k)} + 2C'_{jk}) \\
&\quad + 2\mathcal{H}^2\delta_{jk}(C^l_i + {\epsilon^l}_{ih}\Omega^h) + 4\mathcal{H}^2\delta^l_iC_{jk}\Big]\,x^i_Fx^j_Fx^k_F
\end{aligned}
\tag{131}
$$

Thus the third order term is

$$
\begin{aligned}
& -\frac{1}{6}\left[\Gamma^l_{\alpha\beta,\gamma} - 2\Gamma^l_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma}\right]_P [e_i]^\alpha_P\,[e_j]^\beta_P\,[e_k]^\gamma_P\,x^i_Fx^j_Fx^k_F \\
&= \frac{1}{6a^3}\Big\{\mathcal{H}^2[-2\delta^l_i\delta_{jk}(1-2A) + 2\delta_{jk}(C^l_i + {\epsilon^l}_{ih}\Omega^h)] \\
&\qquad + \mathcal{H}[\delta_{jk}(-{B_i}^{,l} + {B^l}_{,i} - 2C^{l\prime}_i) - \delta^l_i(2B_{(j,k)} + 2C'_{jk}) + \delta_{ij}{B^l}_{,k}] + C^l_{i,jk} + C^l_{j,ik} - {C_{ij,k}}^{,l}\Big\} \\
&\qquad x^i_Fx^j_Fx^k_F \\
&= \frac{1}{6a^3}\Big\{\mathcal{H}^2[-2\delta^l_i\delta_{jk}(1-2A) + 2\delta_{jk}(C^l_i + {\epsilon^l}_{ih}\Omega^h)] \\
&\qquad + \mathcal{H}[\delta_{jk}(-{B_i}^{,l} + 2{B^l}_{,i} - 2C^{l\prime}_i) - \delta^l_i(2B_{(j,k)} + 2C'_{jk})] + C^l_{i,jk} + C^l_{j,ik} - {C_{ij,k}}^{,l}\Big\}\,x^i_Fx^j_Fx^k_F
\end{aligned}
\tag{132}
$$

Finally we get:

$$\begin{aligned} x^l_Q =& x^l_P + \frac{1}{a}(\delta^l_i - C^l_i - \epsilon^l{}_{ih}\Omega^h)x^i_F - \frac{1}{2a^2}\left[2\mathcal{H}\delta^l_i(V_j - B_j) + \mathcal{H}\delta_{ij}B^l + C^l_{i,j} + C^l_{i,j} - C_{ij}{}^{,l}\right]x^i_F x^j_F \\ &+ \frac{1}{6a^3}\Big\{\mathcal{H}^2[-2\delta^l_i\delta_{jk}(1-2A) + 2\delta_{jk}(C^l_i + \epsilon^l{}_{ih}\Omega^h)] \\ &\quad + \mathcal{H}[\delta_{jk}(-B_i{}^{,l} + 2B^l{}_{,i} - 2C^{l\prime}_i) - \delta^l_i(2B_{(j,k)} + 2C'_{jk})] + C^l_{i,jk} + C^l_{j,ik} - C_{ij,k}{}^{,l}\Big\} x^i_F x^j_F x^k_F \end{aligned} \tag{133}$$

We have calculated the Taylor expansion of the coordinates of a point Q near point P on a timelike geodesic. If we set all the perturbation terms to 0 then the results are reduced to the non-perturbed Robertson-Walker space-time case.

8.2 Derivative terms $\frac{\partial x^\mu}{\partial x^\alpha_F}(Q)$

To compute the metric in the FNC, we need to compute the derivatives of the coordinate transformation $g^F_{\mu\nu}(Q) = \frac{\partial x^\alpha}{\partial x^\mu_F}\frac{\partial x^\beta}{\partial x^\nu_F}g_{\alpha\beta}(Q)$. We have calculated all the derivative terms in section 7. Again we bring in the tetrads and Christoffel symbols based on the perturbed Robertson-Walker metric. The 00 component is

$$\frac{\partial x^0}{\partial x^0_F}(Q) = [e_t]^0_P - \Gamma^0_{\alpha\beta}\big|_P [e_i]^\alpha_P [e_t]^\beta_P x^i_F - \frac{1}{2}\left[\Gamma^0_{\alpha\beta,\gamma} - 2\Gamma^0_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P [e_t]^\gamma_P [e_i]^\alpha_P [e_j]^\beta_P x^i_F x^j_F \tag{134}$$

The zeroth order term is

$$[e_t]^0 = \frac{1}{a}(1 - A) \tag{135}$$

The first order term is

$$\begin{aligned} \Gamma^0_{\alpha\beta}[e_i]^\alpha[e_t]^\beta x^i_F &= \left\{\Gamma^0_{00}[e_i]^0[e_t]^0 + \Gamma^0_{a0}[e_i]^a[e_t]^0 + \Gamma^0_{0b}[e_i]^0[e_t]^b + \Gamma^0_{ab}[e_i]^a[e_t]^b\right\}x^i_F \\ &= \frac{1}{a^2}\left[\mathcal{H}(V_i - B_i) + (A_{,a} - \mathcal{H}B_a)\delta^a_i + \mathcal{H}\delta_{ab}\delta^a_i V^b\right]x^i_F \\ &= \frac{1}{a^2}\left[\mathcal{H}(V_i - B_i) + A_{,i} - \mathcal{H}B_i + \mathcal{H}V_i\right]x^i_F \\ &= \frac{1}{a^2}\left[2\mathcal{H}(V_i - B_i) + A_{,i}\right]x^i_F \end{aligned} \tag{136}$$

Calculate the second order term by parts, the first term is

$$
\begin{aligned}
&\Gamma^0_{\alpha\beta,\gamma}\,[e_t]^\gamma\,[e_i]^\alpha\,[e_j]^\beta\,x^i_F x^j_F \\
&= \Big\{\Gamma^0_{00,0}\,[e_t]^0\,[e_i]^0\,[e_j]^0 + \Gamma^0_{a0,0}\,[e_t]^0\,[e_i]^a\,[e_j]^0 + \Gamma^0_{0b,0}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^0_{00,c}\,[e_t]^c\,[e_i]^0\,[e_j]^0 \\
&\quad + \Gamma^0_{ab,0}\,[e_t]^0\,[e_i]^a\,[e_j]^b + \Gamma^0_{a0,c}\,[e_t]^c\,[e_i]^a\,[e_j]^0 + \Gamma^0_{0b,c}\,[e_t]^c\,[e_i]^0\,[e_j]^b + \Gamma^0_{ab,c}\,[e_t]^c\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big\{\Gamma^0_{ab,0}\,[e_t]^0\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[\mathcal{H}'\delta_{ab} - 2\mathcal{H}'\delta_{ab}A - 2\mathcal{H}\delta_{ab}A' + B'_{(a,b)} + C''_{ab} + 2\mathcal{H}'C_{ab} + 2\mathcal{H}C'_{ab}\Big] \\
&\qquad (1-A)(\delta^a_i - C^a_i - \epsilon^a{}_{ih}\Omega^h)(\delta^b_j - C^b_j - \epsilon^b{}_{jh}\Omega^h)x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[\mathcal{H}'\delta_{ab}\delta^a_i\delta^b_j + (-2\mathcal{H}'\delta_{ab}A - 2\mathcal{H}\delta_{ab}A' + B'_{(a,b)} + C''_{ab} + 2\mathcal{H}'C_{ab} + 2\mathcal{H}C'_{ab})\delta^a_i\delta^b_j \\
&\qquad + \mathcal{H}'\delta_{ab}(-A)\delta^a_i\delta^b_j + \mathcal{H}'\delta_{ab}(-C^a_i - \epsilon^a{}_{ih}\Omega^h)\delta^b_j + \mathcal{H}'\delta_{ab}\delta^a_i(-C^b_j - \epsilon^b{}_{jh}\Omega^h)\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[\mathcal{H}'\delta_{ij} - 2\mathcal{H}'\delta_{ij}A - 2\mathcal{H}\delta_{ij}A' + B'_{(i,j)} + C''_{ij} + 2\mathcal{H}'C_{ij} + 2\mathcal{H}C'_{ij} - \mathcal{H}'\delta_{ij}A \\
&\qquad - \mathcal{H}'C_{ij} - \mathcal{H}'\epsilon_{jih}\Omega^h - \mathcal{H}'C_{ij} - \mathcal{H}'\epsilon_{ijh}\Omega^h\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[\mathcal{H}'\delta_{ij} - 3\mathcal{H}'\delta_{ij}A - 2\mathcal{H}\delta_{ij}A' + 2\mathcal{H}C'_{ij} + B'_{(i,j)} + C''_{ij}\Big]\,x^i_F x^j_F
\end{aligned} \tag{137}
$$

and the second term is

$$
\begin{aligned}
&-2\Gamma^0_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\,[e_t]^\gamma\,[e_i]^\alpha\,[e_j]^\beta\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^0_{\sigma0}\Gamma^\sigma_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^0 + \Gamma^0_{\sigma b}\Gamma^\sigma_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^0_{\sigma0}\Gamma^\sigma_{c0}\,[e_t]^c\,[e_i]^0\,[e_j]^0 \\
&\quad + \Gamma^0_{\sigma0}\Gamma^\sigma_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 + \Gamma^0_{\sigma b}\Gamma^\sigma_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b + \Gamma^0_{\sigma b}\Gamma^\sigma_{c0}\,[e_t]^c\,[e_i]^0\,[e_j]^b \\
&\quad + \Gamma^0_{\sigma0}\Gamma^\sigma_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^0 + \Gamma^0_{\sigma b}\Gamma^\sigma_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^0_{0b}\Gamma^0_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^0_{db}\Gamma^d_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^0_{00}\Gamma^0_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 \\
&\quad + \Gamma^0_{d0}\Gamma^d_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 + \Gamma^0_{0b}\Gamma^0_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b + \Gamma^0_{db}\Gamma^d_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b \\
&\quad + \Gamma^0_{0b}\Gamma^0_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b + \Gamma^0_{db}\Gamma^d_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \\
&= -\frac{2}{a^3}\Big[(\mathcal{H}\delta_{db} - 2\mathcal{H}\delta_{db}A + B_{(d,b)} + C'_{db} + 2\mathcal{H}C_{db})(\mathcal{H}\delta^d_a + \frac{1}{2}B_a{}^{,d} - \frac{1}{2}B^d{}_{,a} + C^{d\prime}_a) \\
&\qquad (1-A)(\delta^a_i - C^a_i - \epsilon^a{}_{ih}\Omega^h)(\delta^b_j - C^b_j - \epsilon^b{}_{jh}\Omega^h)\Big]\,x^i_F x^j_F \\
&= -\frac{2}{a^3}\Big[\mathcal{H}^2\delta_{db}\delta^d_a\delta^a_i\delta^b_j + (-2\mathcal{H}\delta_{db}A + B_{(d,b)} + C'_{db} + 2\mathcal{H}C_{db})\mathcal{H}\delta^d_a\delta^a_i\delta^b_j \\
&\qquad + \mathcal{H}\delta_{db}(\frac{1}{2}B_a{}^{,d} - \frac{1}{2}B^d{}_{,a} + C^{d\prime}_a)\delta^a_i\delta^b_j - \mathcal{H}^2\delta_{db}\delta^d_a\delta^a_i\delta^b_j A + \mathcal{H}^2\delta_{db}\delta^d_a(-C^a_i - \epsilon^a{}_{ih}\Omega^h)\delta^b_j \\
&\qquad + \mathcal{H}^2\delta_{db}\delta^d_a\delta^a_i(-C^b_j - \epsilon^b{}_{jh}\Omega^h)\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\big[-2\mathcal{H}^2\delta_{ij} + 4\mathcal{H}^2\delta_{ij}A - 2\mathcal{H}B_{(i,j)} - 2\mathcal{H}C'_{ij} - 4\mathcal{H}^2C_{ij} - \mathcal{H}B_{i,j} + \mathcal{H}B_{j,i} - 2\mathcal{H}C'_{ij} \\
&\qquad + 2\mathcal{H}^2\delta_{ij}A + 2\mathcal{H}^2C_{ij} + 2\mathcal{H}^2\epsilon_{jih}\Omega^h + 2\mathcal{H}^2C_{ij} + 2\mathcal{H}^2\epsilon_{ijh}\Omega^h\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\big[-2\mathcal{H}^2\delta_{ij} + 6\mathcal{H}^2\delta_{ij}A - 2\mathcal{H}B_{i,j} - 4\mathcal{H}C'_{ij}\big]\,x^i_F x^j_F
\end{aligned} \tag{138}
$$

Finally we get:

$$\begin{aligned}\frac{\partial x^0}{\partial x_F^0}(Q) &= \frac{1}{a}(1-A) - \frac{1}{a^2}\left[2\mathcal{H}(V_i - B_i) + A_{,i}\right] x_F^i \\ &\quad - \frac{1}{a^3}[\mathcal{H}'\delta_{ij} - 3\mathcal{H}'\delta_{ij}A - 2\mathcal{H}\delta_{ij}A' - 2\mathcal{H}C'_{ij} + B'_{(i,j)} + C''_{ij} - 2\mathcal{H}^2\delta_{ij} + 6\mathcal{H}^2\delta_{ij}A \\ &\qquad - 2\mathcal{H}B_{i,j}]x_F^i x_F^j \\ &= \frac{1}{a}(1-A) - \frac{1}{a^2}\left[2\mathcal{H}(V_i - B_i) + A_{,i}\right] x_F^i \\ &\quad - \frac{1}{a^3}[\mathcal{H}'(\delta_{ij} - 3\delta_{ij}A) - 2\mathcal{H}^2(\delta_{ij} - 3\delta_{ij}A) - \mathcal{H}(2\delta_{ij}A' + 2C'_{ij} + 2B_{i,j}) \\ &\qquad + B'_{(i,j)} + C''_{ij}]x_F^i x_F^j\end{aligned} \tag{139}$$

The $k0$ component is

$$\frac{\partial x^k}{\partial x_F^0}(Q) = [e_t]_P^k - \Gamma^k_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_t]_P^\beta x_F^i - \frac{1}{2}\left[\Gamma^k_{\alpha\beta,\gamma} - 2\Gamma^k_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P [e_t]_P^\gamma [e_i]_P^\alpha [e_j]_P^\beta x_F^i x_F^j \tag{140}$$

The zeroth order term is

$$[e_t]^k = \frac{1}{a}V^k \tag{141}$$

The first order term is

$$\begin{aligned}\Gamma^k_{\alpha\beta}[e_i]^\alpha [e_t]^\beta x_F^i &= \left\{\Gamma^k_{00}[e_i]^0[e_t]^0 + \Gamma^k_{a0}[e_i]^a[e_t]^0 + \Gamma^k_{0b}[e_i]^0[e_t]^b + \Gamma^k_{ab}[e_i]^a[e_t]^b\right\} x_F^i \\ &= \frac{1}{a^2}\left[(\mathcal{H}\delta^k_a + \frac{1}{2}B_a{}^{,k} - \frac{1}{2}B^k{}_{,a} + C^{k\prime}_a)(\delta^a_i - C^a_i - \epsilon^a{}_{ih}\Omega^h)(1-A)\right] x_F^i \\ &= \frac{1}{a^2}\left[\mathcal{H}\delta^k_i + \frac{1}{2}B_i{}^{,k} - \frac{1}{2}B^k{}_{,i} + C^{k\prime}_i - \mathcal{H}C^k_i - \mathcal{H}\epsilon^k{}_{ih}\Omega^h - \mathcal{H}\delta^k_i A\right] x_F^i \\ &= \frac{1}{a^2}\left[\mathcal{H}(\delta^k_i - \delta^k_i A - C^k_i - \epsilon^k{}_{ih}\Omega^h) + \frac{1}{2}B_i{}^{,k} - \frac{1}{2}B^k{}_{,i} + C^{k\prime}_i\right] x_F^i\end{aligned} \tag{142}$$

Calculate the second order term by parts, the first term is

$$\begin{aligned}&\Gamma^k_{\alpha\beta,\gamma}[e_t]^\gamma [e_i]^\alpha [e_j]^\beta x_F^i x_F^j \\ &= \Big\{\Gamma^k_{00,0}[e_t]^0[e_i]^0[e_j]^0 + \Gamma^k_{a0,0}[e_t]^0[e_i]^a[e_j]^0 + \Gamma^k_{0b,0}[e_t]^0[e_i]^0[e_j]^b + \Gamma^k_{00,c}[e_t]^c[e_i]^0[e_j]^0 \\ &\quad + \Gamma^k_{ab,0}[e_t]^0[e_i]^a[e_j]^b + \Gamma^k_{a0,c}[e_t]^c[e_i]^a[e_j]^0 + \Gamma^k_{0b,c}[e_t]^c[e_i]^0[e_j]^b + \Gamma^k_{ab,c}[e_t]^c[e_i]^a[e_j]^b\Big\} x_F^i x_F^j \\ &= \left\{\Gamma^k_{a0,0}[e_t]^0[e_i]^a[e_j]^0 + \Gamma^k_{0b,0}[e_t]^0[e_i]^0[e_j]^b + \Gamma^k_{ab,0}[e_t]^0[e_i]^a[e_j]^b\right\} x_F^i x_F^j \\ &= \frac{1}{a^3}\Big[\mathcal{H}'\delta^k_a\delta^a_i(V_j - B_j) + \mathcal{H}'\delta^k_b\delta^b_j(V_i - B_i) \\ &\qquad + (\mathcal{H}'\delta_{ab}B^k + \mathcal{H}\delta_{ab}B^{k\prime} + C^{k\prime}_{a,b} + C^{k\prime}_{b,a} - C'_{ab}{}^{,k})\delta^a_i\delta^b_j\Big] x_F^i x_F^j \\ &= \frac{1}{a^3}\left[2\mathcal{H}'\delta^k_i(V_j - B_j) + \mathcal{H}'\delta_{ij}B^k + \mathcal{H}\delta_{ij}B^{k\prime} + C^{k\prime}_{i,j} + C^{k\prime}_{j,i} - C'_{ij}{}^{,k}\right] x_F^i x_F^j\end{aligned} \tag{143}$$

and the second term is

$$
\begin{aligned}
&-2\Gamma^k_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\,[e_t]^\gamma\,[e_i]^\alpha\,[e_j]^\beta\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^k_{\sigma 0}\Gamma^\sigma_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^0 + \Gamma^k_{\sigma b}\Gamma^\sigma_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^k_{\sigma 0}\Gamma^\sigma_{c0}\,[e_t]^c\,[e_i]^0\,[e_j]^0 \\
&\qquad +\Gamma^k_{\sigma 0}\Gamma^\sigma_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 + \Gamma^k_{\sigma b}\Gamma^\sigma_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b + \Gamma^k_{\sigma b}\Gamma^\sigma_{c0}\,[e_t]^c\,[e_i]^0\,[e_j]^b \\
&\qquad +\Gamma^k_{\sigma 0}\Gamma^\sigma_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^0 + \Gamma^k_{\sigma b}\Gamma^\sigma_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^k_{0b}\Gamma^0_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^k_{db}\Gamma^d_{00}\,[e_t]^0\,[e_i]^0\,[e_j]^b + \Gamma^k_{00}\Gamma^0_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 \\
&\qquad +\Gamma^k_{d0}\Gamma^d_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^0 + \Gamma^k_{0b}\Gamma^0_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b + \Gamma^k_{db}\Gamma^d_{0a}\,[e_t]^0\,[e_i]^a\,[e_j]^b \\
&\qquad +\Gamma^k_{0b}\Gamma^0_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b + \Gamma^k_{db}\Gamma^d_{ca}\,[e_t]^c\,[e_i]^a\,[e_j]^b\Big\}\,x^i_F x^j_F \qquad (144)\\
&= -2\frac{1}{a^3}\Big[\mathcal{H}\delta^k_b\mathcal{H}(V_i - B_i)\delta^b_j + \mathcal{H}\delta^k_d\mathcal{H}\delta^d_a\delta^a_i(V_j - B_j) + \mathcal{H}\delta^k_b(A_{,a} - \mathcal{H}B_a)\delta^a_i\delta^b_j \\
&\qquad +\mathcal{H}\delta^d_a(\mathcal{H}\delta_{db}B^k + C^k_{d,b} + C^k_{b,d} - C_{db}{}^{,k})\delta^a_i\delta^b_j + \mathcal{H}\delta^k_b\mathcal{H}\delta_{ca}V^c\delta^a_i\delta^b_j\Big]\,x^i_F x^j_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}^2\delta^k_j(V_i - B_i) + \mathcal{H}^2\delta^k_i(V_j - B_j) + \mathcal{H}\delta^k_j A_{,i} - \mathcal{H}^2\delta^k_j B_i + \mathcal{H}^2\delta_{ij}B^k + \mathcal{H}C^k_{i,j} \\
&\qquad +\mathcal{H}C^k_{j,i} - \mathcal{H}C_{ij}{}^{,k} + \mathcal{H}^2\delta^k_j V_i\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[-6\mathcal{H}^2\delta^k_i(V_j - B_j) - 2\mathcal{H}^2\delta_{ij}B^k - 2\mathcal{H}(\delta^k_j A_{,i} + 2C^k_{(i,j)} - C_{ij}{}^{,k})\Big]\,x^i_F x^j_F
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&-\frac{1}{2}\Big[\Gamma^k_{\alpha\beta,\gamma} - 2\Gamma^k_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\Big]_P\,[e_t]^\gamma_P\,[e_i]^\alpha_P\,[e_j]^\beta_P\,x^i_F x^j_F \\
&= -\frac{1}{2a^3}\Big[2\mathcal{H}'\delta^k_i(V_j - B_j) + \mathcal{H}'\delta_{ij}B^k - 6\mathcal{H}^2\delta^k_i(V_j - B_j) - 2\mathcal{H}^2\delta_{ij}B^k \qquad (145)\\
&\qquad -\mathcal{H}(2\delta^k_j A_{,i} + 4C^k_{(i,j)} - 2C_{ij}{}^{,k} - \delta_{ij}B^{k\prime}) + 2C^{k\prime}_{(i,j)} - C'_{ij}{}^{,k}\Big]\,x^i_F x^j_F
\end{aligned}
$$

Finally we get:

$$
\begin{aligned}
\frac{\partial x^k}{\partial x^0_F}(Q) &= \frac{1}{a}V^k - \frac{1}{a^2}\Big[\mathcal{H}(\delta^k_i - \delta^k_i A - C^k_i - \epsilon^k{}_{ih}\Omega^h) + \frac{1}{2}B_i{}^{,k} - \frac{1}{2}B^k{}_{,i} + C^{k\prime}_i\Big]\,x^i_F \\
&\quad - \frac{1}{2a^3}\Big[2\mathcal{H}'\delta^k_i(V_j - B_j) + \mathcal{H}'\delta_{ij}B^k - 6\mathcal{H}^2\delta^k_i(V_j - B_j) - 2\mathcal{H}^2\delta_{ij}B^k \qquad (146)\\
&\qquad -\mathcal{H}(2\delta^k_j A_{,i} + 4C^k_{(i,j)} - 2C_{ij}{}^{,k} - \delta_{ij}B^{k\prime}) + 2C^{k\prime}_{(i,j)} - C'_{ij}{}^{,k}\Big]\,x^i_F x^j_F
\end{aligned}
$$

The $0l$ component is

$$
\begin{aligned}
\frac{\partial x^0}{\partial x^l_F}(Q) &= [e_l]^0_P - \Gamma^0_{\alpha\beta}\big|_P\,[e_i]^\alpha_P\,[e_l]^\beta_P\,x^i_F \\
&\quad - \frac{1}{6}\left[\Gamma^0_{\alpha\beta,\gamma} + 2\Gamma^0_{\gamma\alpha,\beta} - 2\Gamma^0_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta} - 4\Gamma^0_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P\,[e_i]^\alpha_P\,[e_j]^\beta_P\,[e_l]^\gamma_P\,x^i_F x^j_F \qquad (147)
\end{aligned}
$$

The first order term is

$$\begin{aligned}
-\Gamma^0_{\alpha\beta}\left[e_i\right]^\alpha\left[e_l\right]^\beta x^i_F &= -\left\{\Gamma^0_{00}\left[e_i\right]^0\left[e_l\right]^0+\Gamma^0_{a0}\left[e_i\right]^a\left[e_l\right]^0+\Gamma^0_{0b}\left[e_i\right]^0\left[e_l\right]^b+\Gamma^0_{ab}\left[e_i\right]^a\left[e_l\right]^b\right\}x^i_F\\
&= -\frac{1}{a^2}\left[(\mathcal{H}\delta_{ab}-2\mathcal{H}\delta_{ab}A+B_{(a,b)}+C'_{ab}+2\mathcal{H}C_{ab})\right.\\
&\qquad\left.(\delta^a_i-C^a_i-\epsilon^a{}_{ih}\Omega^h)(\delta^b_l-C^b_l-\epsilon^b{}_{lh}\Omega^h)\right]x^i_F\\
&= -\frac{1}{a^2}\left[(\mathcal{H}\delta_{ab}-2\mathcal{H}\delta_{ab}A+B_{(a,b)}+C'_{ab}+2\mathcal{H}C_{ab})\delta^a_i\delta^b_l\right.\\
&\qquad\left.+\mathcal{H}\delta_{ab}\delta^a_i(-C^b_l-\epsilon^b{}_{lh}\Omega^h)+\mathcal{H}\delta_{ab}\delta^b_l(-C^a_i-\epsilon^a{}_{ih}\Omega^h)\right]x^i_F\\
&= -\frac{1}{a^2}\left[\mathcal{H}\delta_{il}-2\mathcal{H}\delta_{il}A+B_{(i,l)}+C'_{il}+2\mathcal{H}C_{il}\right.\\
&\qquad\left.+\mathcal{H}(-C_{il}-\epsilon_{ilh}\Omega^h)+\mathcal{H}(-C_{il}-\epsilon_{lih}\Omega^h)\right]x^i_F\\
&= -\frac{1}{a^2}\left[\mathcal{H}\delta_{il}-2\mathcal{H}\delta_{il}A+B_{(i,l)}+C'_{il}\right]x^i_F
\end{aligned} \tag{148}$$

Calculate the second order term by parts, the first term is

$$\begin{aligned}
&\Gamma^0_{\alpha\beta,\gamma}\left[e_i\right]^\alpha\left[e_j\right]^\beta\left[e_l\right]^\gamma x^i_Fx^j_F\\
&=\left\{\Gamma^0_{00,0}\left[e_i\right]^0\left[e_j\right]^0\left[e_l\right]^0+\Gamma^0_{a0,0}\left[e_i\right]^a\left[e_j\right]^0\left[e_l\right]^0+\Gamma^0_{0b,0}\left[e_i\right]^0\left[e_j\right]^b\left[e_l\right]^0+\Gamma^0_{00,c}\left[e_i\right]^0\left[e_j\right]^0\left[e_l\right]^c\right.\\
&\qquad\left.+\Gamma^0_{ab,0}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^0+\Gamma^0_{a0,c}\left[e_i\right]^a\left[e_j\right]^0\left[e_l\right]^c+\Gamma^0_{0b,c}\left[e_i\right]^0\left[e_j\right]^b\left[e_l\right]^c+\Gamma^0_{ab,c}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^c\right\}x^i_Fx^j_F\\
&=\left\{\Gamma^0_{ab,0}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^0+\Gamma^0_{ab,c}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^c\right\}x^i_Fx^j_F\\
&=\frac{1}{a^3}\left[\mathcal{H}'\delta_{ab}\delta^a_i\delta^b_j(V_l-B_l)+(-2\mathcal{H}\delta_{ab}A_{,c}+B_{(a,b),c}+C'_{ab,c}+2\mathcal{H}C_{ab,c})\delta^a_i\delta^b_j\delta^c_l\right]x^i_Fx^j_F\\
&=\frac{1}{a^3}\left[\mathcal{H}'\delta_{ij}(V_l-B_l)-2\mathcal{H}\delta_{ij}A_{,l}+B_{(i,j),l}+C'_{ij,l}+2\mathcal{H}C_{ij,l}\right]x^i_Fx^j_F,
\end{aligned} \tag{149}$$

The second term is

$$\begin{aligned}
&2\Gamma^0_{\gamma\alpha,\beta}\left[e_i\right]^\alpha\left[e_j\right]^\beta\left[e_l\right]^\gamma x^i_Fx^j_F\\
&=2\left\{\Gamma^0_{00,0}\left[e_i\right]^0\left[e_j\right]^0\left[e_l\right]^0+\Gamma^0_{0a,0}\left[e_i\right]^a\left[e_j\right]^0\left[e_l\right]^0+\Gamma^0_{00,b}\left[e_i\right]^0\left[e_j\right]^b\left[e_l\right]^0+\Gamma^0_{c0,0}\left[e_i\right]^0\left[e_j\right]^0\left[e_l\right]^c\right.\\
&\qquad\left.+\Gamma^0_{0a,b}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^0+\Gamma^0_{ca,0}\left[e_i\right]^a\left[e_j\right]^0\left[e_l\right]^c+\Gamma^0_{c0,b}\left[e_i\right]^0\left[e_j\right]^b\left[e_l\right]^c+\Gamma^0_{ca,b}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^c\right\}x^i_Fx^j_F\\
&=2\left\{\Gamma^0_{ca,0}\left[e_i\right]^a\left[e_j\right]^0\left[e_l\right]^c+\Gamma^0_{ca,b}\left[e_i\right]^a\left[e_j\right]^b\left[e_l\right]^c\right\}x^i_Fx^j_F\\
&=2\frac{1}{a^3}\left[\mathcal{H}'\delta_{ca}\delta^a_i\delta^c_l(V_j-B_j)+(-2\mathcal{H}\delta_{ca}A_{,b}+B_{(c,a),b}+C'_{ca,b}+2\mathcal{H}C_{ca,b})\delta^a_i\delta^b_j\delta^c_l\right]x^i_Fx^j_F\\
&=\frac{1}{a^3}\left[2\mathcal{H}'\delta_{li}(V_j-B_j)-4\mathcal{H}\delta_{li}A_{,j}+2B_{(l,i),j}+2C'_{li,j}+4\mathcal{H}C_{li,j}\right]x^i_Fx^j_F
\end{aligned} \tag{150}$$

The third term is

$$
\begin{aligned}
&-2\Gamma^0_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta}\,[e_i]^\alpha\,[e_j]^\beta\,[e_l]^\gamma\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^0_{\sigma 0}\Gamma^\sigma_{00}\,[e_i]^0\,[e_j]^0\,[e_l]^0 + \Gamma^0_{\sigma 0}\Gamma^\sigma_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^0 + \Gamma^0_{\sigma 0}\Gamma^\sigma_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^0 \\
&\quad +\Gamma^0_{\sigma c}\Gamma^\sigma_{00}\,[e_i]^0\,[e_j]^0\,[e_l]^c + \Gamma^0_{\sigma 0}\Gamma^\sigma_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^0_{\sigma c}\Gamma^\sigma_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c \\
&\quad +\Gamma^0_{\sigma c}\Gamma^\sigma_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c + \Gamma^0_{\sigma c}\Gamma^\sigma_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^0_{00}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^0_{d0}\Gamma^d_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^0_{0c}\Gamma^0_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c \\
&\quad +\Gamma^0_{dc}\Gamma^d_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c + \Gamma^0_{0c}\Gamma^0_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c + \Gamma^0_{dc}\Gamma^d_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c \\
&\quad +\Gamma^0_{0c}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c + \Gamma^0_{dc}\Gamma^d_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^0_{00}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^0_{dc}\Gamma^d_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c + \Gamma^0_{dc}\Gamma^d_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c \\
&\quad +\Gamma^0_{0c}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c + \Gamma^0_{dc}\Gamma^d_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}\mathcal{H}\delta_{ab}\delta^a_i\delta^b_j(V_l - B_l) + \mathcal{H}\delta_{dc}\mathcal{H}\delta^d_a\delta^a_i\delta^c_l(V_j - B_j) + \mathcal{H}\delta_{dc}\mathcal{H}\delta^d_b\delta^b_j\delta^c_l(V_i - B_i) \\
&\qquad +(A_{,c} - \mathcal{H}B_c)\mathcal{H}\delta_{ab}\delta^a_i\delta^b_j\delta^c_l + \mathcal{H}\delta_{dc}(\mathcal{H}\delta_{ab}B^d + C^d_{a,b} + C^d_{b,a} - C_{ab}{}^{,d})\delta^a_i\delta^b_j\delta^c_l\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\left[-2\mathcal{H}^2\delta_{ij}(V_l - B_l) - 2\mathcal{H}^2\delta_{il}(V_j - B_j) - 2\mathcal{H}^2\delta_{jl}(V_i - B_i)\right. \\
&\quad \left.-2\mathcal{H}\delta_{ij}A_{,l} - 2\mathcal{H}C_{li,j} - 2\mathcal{H}C_{lj,i} + 2\mathcal{H}C_{ij,l}\right]x^i_F x^j_F
\end{aligned} \tag{151}
$$

and the forth term is

$$
\begin{aligned}
-4\Gamma^0_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\,[e_i]^\alpha\,[e_j]^\beta\,[e_l]^\gamma\,x^i_F x^j_F = \frac{1}{a^3}&\left[-4\mathcal{H}^2\delta_{il}(V_j - B_j) - 4\mathcal{H}^2\delta_{ij}(V_l - B_l) - 4\mathcal{H}^2\delta_{jl}(V_i - B_i)\right. \\
&\left.-4\mathcal{H}\delta_{il}A_{,j} - 4\mathcal{H}C_{ji,l} - 4\mathcal{H}C_{lj,i} + 4\mathcal{H}C_{il,j}\right]x^i_F x^j_F
\end{aligned} \tag{152}
$$

Finally we get:

$$
\begin{aligned}
\frac{\partial x^0}{\partial x^l_F}(Q) &= \frac{1}{a}(V_l - B_l) - \frac{1}{a^2}\left[\mathcal{H}\delta_{il} - 2\mathcal{H}\delta_{il}A + B_{(i,l)} + C'_{il}\right]x^i_F \\
&\quad -\frac{1}{6a^3}\left[\mathcal{H}'\delta_{ij}(V_l - B_l) - 2\mathcal{H}\delta_{ij}A_{,l} + B_{(i,j),l} + C'_{ij,l} + 2\mathcal{H}C_{ij,l} + 2\mathcal{H}'\delta_{li}(V_j - B_j)\right. \\
&\qquad -4\mathcal{H}\delta_{li}A_{,j} + 2B_{(l,i),j} + 2C'_{li,j} + 4\mathcal{H}C_{li,j} - 2\mathcal{H}^2\delta_{ij}(V_l - B_l) \\
&\qquad -2\mathcal{H}^2\delta_{il}(V_j - B_j) - 2\mathcal{H}^2\delta_{jl}(V_i - B_i) - 2\mathcal{H}\delta_{ij}A_{,l} - 2\mathcal{H}C_{li,j} - 2\mathcal{H}C_{lj,i} \\
&\qquad +2\mathcal{H}C_{ij,l} - 4\mathcal{H}^2\delta_{il}(V_j - B_j) - 4\mathcal{H}^2\delta_{ij}(V_l - B_l) - 4\mathcal{H}^2\delta_{jl}(V_i - B_i) \\
&\qquad \left.-4\mathcal{H}\delta_{il}A_{,j} - 4\mathcal{H}C_{ji,l} - 4\mathcal{H}C_{lj,i} + 4\mathcal{H}C_{il,j}\right]x^i_F x^j_F \\
&= \frac{1}{a}(V_l - B_l) - \frac{1}{a^2}\left[\mathcal{H}\delta_{il} - 2\mathcal{H}\delta_{il}A + B_{(i,l)} + C'_{il}\right]x^i_F \\
&\quad -\frac{1}{6a^3}\left[(\mathcal{H}' - 6\mathcal{H}^2)\delta_{ij}(V_l - B_l) + 2(\mathcal{H}' - 6\mathcal{H}^2)\delta_{li}(V_j - B_j) - 4\mathcal{H}\delta_{ij}A_{,l} - 8\mathcal{H}\delta_{li}A_{,j}\right. \\
&\qquad \left.+B_{(i,j),l} + C'_{ij,l} + 2B_{(l,i),j} + 2C'_{li,j}\right]x^i_F x^j_F
\end{aligned} \tag{153}
$$

The kl component is

$$\frac{\partial x^k}{\partial x_F^l}(Q) = [e_l]_P^k - \Gamma^k_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_l]_P^\beta x_F^i - \frac{1}{6}\left[\Gamma^k_{\alpha\beta,\gamma} + 2\Gamma^k_{\gamma\alpha,\beta} - 2\Gamma^k_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta} - 4\Gamma^k_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\right]_P [e_i]_P^\alpha [e_j]_P^\beta [e_l]_P^\gamma x_F^i x_F^j \tag{154}$$

The zeroth order term is

$$[e_l]^k = \frac{1}{a}(\delta_l^k - C_l^k - \epsilon^k{}_{lh}\Omega^h) \tag{155}$$

The first order term is

$$\begin{aligned} - \Gamma^k_{\alpha\beta}\Big|_P [e_i]_P^\alpha [e_l]_P^\beta x_F^i &= -\left\{\Gamma^k_{00}[e_i]^0[e_l]^0 + \Gamma^k_{a0}[e_i]^a[e_l]^0 + \Gamma^k_{0b}[e_i]^0[e_l]^b + \Gamma^k_{ab}[e_i]^a[e_l]^b\right\} x_F^i \\ &= -\frac{1}{a^2}\Big[\mathcal{H}\delta_a^k\delta_i^a(V_l - B_l) + \mathcal{H}\delta_b^k\delta_l^b(V_i - B_i) \\ &\qquad +(\mathcal{H}\delta_{ab}B^k + C^k_{a,b} + C^k_{b,a} - C_{ab}{}^{,k})\delta_i^a\delta_l^b\Big] x_F^i \\ &= -\frac{1}{a^2}\Big[\mathcal{H}\delta_i^k(V_l - B_l) + \mathcal{H}\delta_l^k(V_i - B_i) + \mathcal{H}\delta_{il}B^k + C^k_{i,l} + C^k_{l,i} \\ &\qquad -C_{il}{}^{,k}\Big] x_F^i \end{aligned} \tag{156}$$

Calculate the second order term by parts, the first term is

$$\begin{aligned} &\Gamma^k_{\alpha\beta,\gamma}[e_i]^\alpha[e_j]^\beta[e_l]^\gamma x_F^i x_F^j \\ &= \Big\{\Gamma^k_{00,0}[e_i]^0[e_j]^0[e_l]^0 + \Gamma^k_{a0,0}[e_i]^a[e_j]^0[e_l]^0 + \Gamma^k_{0b,0}[e_i]^0[e_j]^b[e_l]^0 + \Gamma^k_{00,c}[e_i]^0[e_j]^0[e_l]^c \\ &\quad +\Gamma^k_{ab,0}[e_i]^a[e_j]^b[e_l]^0 + \Gamma^k_{a0,c}[e_i]^a[e_j]^0[e_l]^c + \Gamma^k_{0b,c}[e_i]^0[e_j]^b[e_l]^c + \Gamma^k_{ab,c}[e_i]^a[e_j]^b[e_l]^c\Big\} x_F^i x_F^j \\ &= \frac{1}{a^3}\left[(\mathcal{H}\delta_{ab}B^k{}_{,c} + C^k_{a,bc} + C^k_{b,ac} - C_{ab,c}{}^{,k})\delta_i^a\delta_j^b\delta_l^c\right] x_F^i x_F^j \\ &= \frac{1}{a^3}\left[\mathcal{H}\delta_{ij}B^k{}_{,l} + 2C^k_{(i,j),l} - C_{ij,l}{}^{,k}\right] x_F^i x_F^j \end{aligned} \tag{157}$$

The second term is

$$2\Gamma^k_{\gamma\alpha,\beta}[e_i]^\alpha[e_j]^\beta[e_l]^\gamma x_F^i x_F^j = \frac{1}{a^3}\left[2\mathcal{H}\delta_{il}B^k{}_{,j} + 4C^k_{(i,l),j} - 2C_{il,j}{}^{,k}\right] x_F^i x_F^j \tag{158}$$

The third term is

$$
\begin{aligned}
& -2\Gamma^k_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta}\,[e_i]^\alpha\,[e_j]^\beta\,[e_l]^\gamma\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^k_{\sigma 0}\Gamma^\sigma_{00}\,[e_i]^0\,[e_j]^0\,[e_l]^0 + \Gamma^k_{\sigma 0}\Gamma^\sigma_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^0 + \Gamma^k_{\sigma 0}\Gamma^\sigma_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^0 \\
&\qquad +\Gamma^k_{\sigma c}\Gamma^\sigma_{00}\,[e_i]^0\,[e_j]^0\,[e_l]^c + \Gamma^k_{\sigma 0}\Gamma^\sigma_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^k_{\sigma c}\Gamma^\sigma_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c \\
&\qquad +\Gamma^k_{\sigma c}\Gamma^\sigma_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c + \Gamma^k_{\sigma c}\Gamma^\sigma_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^k_{00}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^k_{d0}\Gamma^d_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^0 + \Gamma^k_{0c}\Gamma^0_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c \\
&\qquad +\Gamma^k_{dc}\Gamma^d_{a0}\,[e_i]^a\,[e_j]^0\,[e_l]^c + \Gamma^k_{0c}\Gamma^0_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c + \Gamma^k_{dc}\Gamma^d_{0b}\,[e_i]^0\,[e_j]^b\,[e_l]^c \\
&\qquad +\Gamma^k_{0c}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c + \Gamma^k_{dc}\Gamma^d_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\Big\{\Gamma^k_{0c}\Gamma^0_{ab}\,[e_i]^a\,[e_j]^b\,[e_l]^c\Big\}\,x^i_F x^j_F \\
&= -2\frac{1}{a^3}\Big[(\mathcal{H}\delta^k_c + \frac{1}{2}\Big(B_c{}^{,k} - B^k{}_{,c}\Big) + C^{k\prime}_c)(\mathcal{H}\delta_{ab} - 2\mathcal{H}\delta_{ab}A + B_{(a,b)} + C'_{ab} + 2\mathcal{H}C_{ab}) \\
&\qquad (\delta^a_i - C^a_i - \epsilon^a{}_{ih}\Omega^h)(\delta^b_j - C^b_j - \epsilon^b{}_{jh}\Omega^h)(\delta^c_l - C^c_l - \epsilon^c{}_{lh}\Omega^h)\Big]\,x^i_F x^j_F \\
&= -2\frac{1}{a^3}\Big[\mathcal{H}^2\delta_{ij}\delta^k_l + (\frac{1}{2}B_l{}^{,k} - \frac{1}{2}B^k{}_{,l} + C^{k\prime}_l)\mathcal{H}\delta_{ij} + \mathcal{H}\delta^k_l(-2\mathcal{H}\delta_{ij}A + B_{(i,j)} + C'_{ij} + 2\mathcal{H}C_{ij}) \\
&\qquad +\mathcal{H}^2\delta^k_l\delta_{aj}(-C^a_i - \epsilon^a{}_{ih}\Omega^h) + \mathcal{H}^2\delta^k_l\delta_{ib}(-C^b_j - \epsilon^b{}_{jh}\Omega^h) + \mathcal{H}^2\delta^k_c\delta_{ij}(-C^c_l - \epsilon^c{}_{lh}\Omega^h)\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[-2\mathcal{H}^2\delta_{ij}\delta^k_l + \mathcal{H}\delta_{ij}(-B_l{}^{,k} + B^k{}_{,l} - 2C^{k\prime}_l) + 4\mathcal{H}^2\delta_{ij}\delta^k_l A - 2\mathcal{H}\delta^k_l(B_{(i,j)} + C'_{ij}) \\
&\qquad -4\mathcal{H}^2\delta^k_l C_{ij} + 4\mathcal{H}^2\delta^k_l C_{ij} + 2\mathcal{H}^2\delta_{ij}C^k_l + 2\mathcal{H}^2\delta_{ij}\epsilon^k{}_{lh}\Omega^h\Big]\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[2\mathcal{H}^2(-\delta_{ij}\delta^k_l + 2\delta_{ij}\delta^k_l A + \delta_{ij}C^k_l + \delta_{ij}\epsilon^k{}_{lh}\Omega^h) + \mathcal{H}\delta_{ij}(-B_l{}^{,k} + B^k{}_{,l} - 2C^{k\prime}_l) \\
&\qquad -2\mathcal{H}\delta^k_l(B_{(i,j)} + C'_{ij})\Big]\,x^i_F x^j_F
\end{aligned}
\tag{159}
$$

and the forth term is

$$
\begin{aligned}
& -4\Gamma^k_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha}\,[e_i]^\alpha\,[e_j]^\beta\,[e_l]^\gamma\,x^i_F x^j_F \\
&= \frac{1}{a^3}\Big[4\mathcal{H}^2(-\delta_{il}\delta^k_j + 2\delta_{il}\delta^k_j A + \delta_{il}C^k_j + \delta_{il}\epsilon^k{}_{jh}\Omega^h) + 2\mathcal{H}\delta_{il}(-B_j{}^{,k} + B^k{}_{,j} - 2C^{k\prime}_j) \\
&\qquad -4\mathcal{H}\delta^k_j(B_{(i,l)} + C'_{il})\Big]\,x^i_F x^j_F
\end{aligned}
\tag{160}
$$

Finally we get:

$$\begin{aligned}
\frac{\partial x^k}{\partial x^l_F}(Q) =& \frac{1}{a}(\delta^k_l - C^k_l - \epsilon^k{}_{lh}\Omega^h) \\
&- \frac{1}{a^2}\left[\mathcal{H}\delta^k_i(V_l - B_l) + \mathcal{H}\delta^k_l(V_i - B_i) + \mathcal{H}\delta_{il}B^k + C^k_{i,l} + C^k_{l,i} - C_{il}{}^{,k}\right] x^i_F \\
&- \frac{1}{6a^3}\Big[\mathcal{H}\delta_{ij}B^k{}_{,l} + 2C^k_{(i,j),l} - C_{ij,l}{}^{,k} + 2\mathcal{H}\delta_{il}B^k{}_{,j} + 4C^k_{(i,l),j} - 2C_{il,j}{}^{,k} \\
&\quad +2\mathcal{H}^2(-\delta_{ij}\delta^k_l + 2\delta_{ij}\delta^k_l A + \delta_{ij}C^k_l + \delta_{ij}\epsilon^k{}_{lh}\Omega^h) + \mathcal{H}\delta_{ij}(-B_l{}^{,k} + B^k{}_{,l} - 2C^{k\prime}_l) \\
&\quad -2\mathcal{H}\delta^k_l(B_{(i,j)} + C'_{ij}) + 4\mathcal{H}^2(-\delta_{il}\delta^k_j + 2\delta_{il}\delta^k_j A + \delta_{il}C^k_j + \delta_{il}\epsilon^k{}_{jh}\Omega^h) \\
&\quad +2\mathcal{H}\delta_{il}(-B_j{}^{,k} + B^k{}_{,j} - 2C^{k\prime}_j) - 4\mathcal{H}\delta^k_j(B_{(i,l)} + C'_{il})\Big] x^i_F x^j_F \\
=& \frac{1}{a}(\delta^k_l - C^k_l - \epsilon^k{}_{lh}\Omega^h) \\
&- \frac{1}{a^2}\left[\mathcal{H}\delta^k_i(V_l - B_l) + \mathcal{H}\delta^k_l(V_i - B_i) + \mathcal{H}\delta_{il}B^k + C^k_{i,l} + C^k_{l,i} - C_{il}{}^{,k}\right] x^i_F \\
&- \frac{1}{6a^3}\Big[2C^k_{(i,j),l} - C_{ij,l}{}^{,k} + 4C^k_{(i,l),j} - 2C_{il,j}{}^{,k} \\
&\quad +2\mathcal{H}^2(-\delta_{ij}\delta^k_l + 2\delta_{ij}\delta^k_l A + \delta_{ij}C^k_l + \delta_{ij}\epsilon^k{}_{lh}\Omega^h) + \mathcal{H}\delta_{ij}(-B_l{}^{,k} + 2B^k{}_{,l} - 2C^{k\prime}_l) \\
&\quad -2\mathcal{H}\delta^k_l(B_{(i,j)} + C'_{ij}) + 4\mathcal{H}^2(-\delta_{il}\delta^k_j + 2\delta_{il}\delta^k_j A + \delta_{il}C^k_j + \delta_{il}\epsilon^k{}_{jh}\Omega^h) \\
&\quad +2\mathcal{H}\delta_{il}(-B_j{}^{,k} + 2B^k{}_{,j} - 2C^{k\prime}_j) - 4\mathcal{H}\delta^k_j(B_{(i,l)} + C'_{il})\Big] x^i_F x^j_F
\end{aligned} \tag{161}$$

We have written down all the components of the derivative $\frac{\partial x^\mu}{\partial x^\alpha_F}(Q)$. The derivative shows the relation between the global coordinates and the Fermi normal coordinates. If we set all the perturbations to 0 the result is the case of non-perturbed Robertson-Walker space-time.

8.3 The Riemann tensor in FNC

The Riemann tensor in FNC is defined as $R^F_{\alpha\beta\gamma\delta} = [e_\alpha]^\mu_P [e_\beta]^\nu_P [e_\gamma]^\kappa_P [e_\delta]^\lambda_P R_{\mu\nu\kappa\lambda}$. Bring in the four tetrads and the Riemann tensor based on the perturbed Robertson-Walker metric. First, we calculate the $0l0m$ component.

$$\begin{aligned}
R^F_{0l0m} &= [e_t]_\mu [e_l]^\nu [e_t]^\kappa [e_m]^\lambda R^\mu{}_{\nu\kappa\lambda} \\
&= [e_t]_0 [e_l]^0 [e_t]^0 [e_m]^0 R^0{}_{000} + [e_t]_a [e_l]^0 [e_t]^0 [e_m]^0 R^a{}_{000} + [e_t]_0 [e_l]^b [e_t]^0 [e_m]^0 R^0{}_{b00} \\
&\quad + [e_t]_0 [e_l]^0 [e_t]^c [e_m]^0 R^0{}_{0c0} + [e_t]_0 [e_l]^0 [e_t]^0 [e_m]^d R^0{}_{00d} + [e_t]_a [e_l]^b [e_t]^0 [e_m]^0 R^a{}_{b00} \\
&\quad + [e_t]_a [e_l]^0 [e_t]^c [e_m]^0 R^a{}_{0c0} + [e_t]_a [e_l]^0 [e_t]^0 [e_m]^d R^a{}_{00d} + [e_t]_0 [e_l]^b [e_t]^c [e_m]^0 R^0{}_{bc0} \\
&\quad + [e_t]_0 [e_l]^b [e_t]^0 [e_m]^d R^0{}_{b0d} + [e_t]_0 [e_l]^0 [e_t]^c [e_m]^d R^0{}_{0cd} + [e_t]_a [e_l]^b [e_t]^c [e_m]^0 R^a{}_{bc0} \\
&\quad + [e_t]_a [e_l]^b [e_t]^0 [e_m]^d R^a{}_{b0d} + [e_t]_a [e_l]^0 [e_t]^c [e_m]^d R^a{}_{0cd} + [e_t]_0 [e_l]^b [e_t]^c [e_m]^d R^0{}_{bcd} \\
&\quad + [e_t]_a [e_l]^b [e_t]^c [e_m]^d R^a{}_{bcd} \\
&= [e_t]_0 [e_l]^b [e_t]^0 [e_m]^d R^0{}_{b0d} \\
&= -\frac{1}{a^2}(\delta^b_l - C^b_l - \epsilon^b{}_{lh}\Omega^h)(\delta^d_m - C^d_m - \epsilon^d{}_{mh}\Omega^h) \\
&\qquad [\mathcal{H}'\delta_{bd} - \left(\mathcal{H}A' + 2\mathcal{H}'A\right)\delta_{bd} - A_{,bd} + B'_{(b,d)} + \mathcal{H}B_{(b,d)} + C''_{bd} + \mathcal{H}C'_{bd} + 2\mathcal{H}'C_{bd}] \\
&= -\frac{1}{a^2}[\mathcal{H}'\delta_{lm} - \left(\mathcal{H}A' + 2\mathcal{H}'A\right)\delta_{lm} - A_{,lm} + B'_{(l,m)} + \mathcal{H}B_{(l,m)} + C''_{lm} + \mathcal{H}C'_{lm} + 2\mathcal{H}'C_{lm} \\
&\qquad + (-C^b_l - \epsilon^b{}_{lh}\Omega^h)\delta^d_m\mathcal{H}'\delta_{bd} + \delta^b_l(-C^d_m - \epsilon^d{}_{mh}\Omega^h)\mathcal{H}'\delta_{bd}] \\
&= \frac{1}{a^2}[-\mathcal{H}'\delta_{lm} + (\mathcal{H}A' + 2\mathcal{H}'A)\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm} - 2\mathcal{H}'C_{lm} \\
&\qquad + \mathcal{H}'(C_{lm} + \epsilon_{mlh}\Omega^h + C_{lm} + \epsilon_{lmh}\Omega^h)] \\
&= \frac{1}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A) + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm}]
\end{aligned} \tag{162}$$

The $0lim$ component is

$$
\begin{aligned}
R^F_{0lim} &= [e_t]_\mu \, [e_l]^\nu \, [e_i]^\kappa \, [e_m]^\lambda \, R^\mu{}_{\nu\kappa\lambda} \\
&= [e_t]_0 \, [e_l]^0 \, [e_i]^0 \, [e_m]^0 \, R^0{}_{000} + [e_t]_a \, [e_l]^0 \, [e_i]^0 \, [e_m]^0 \, R^a{}_{000} + [e_t]_0 \, [e_l]^b \, [e_i]^0 \, [e_m]^0 \, R^0{}_{b00} \\
&\quad + [e_t]_0 \, [e_l]^0 \, [e_i]^c \, [e_m]^0 \, R^0{}_{0c0} + [e_t]_0 \, [e_l]^0 \, [e_i]^0 \, [e_m]^d \, R^0{}_{00d} + [e_t]_a \, [e_l]^b \, [e_i]^0 \, [e_m]^0 \, R^a{}_{b00} \\
&\quad + [e_t]_a \, [e_l]^0 \, [e_i]^c \, [e_m]^0 \, R^a{}_{0c0} + [e_t]_a \, [e_l]^0 \, [e_i]^0 \, [e_m]^d \, R^a{}_{00d} + [e_t]_0 \, [e_l]^b \, [e_i]^c \, [e_m]^0 \, R^0{}_{bc0} \\
&\quad + [e_t]_0 \, [e_l]^b \, [e_i]^0 \, [e_m]^d \, R^0{}_{b0d} + [e_t]_0 \, [e_l]^0 \, [e_i]^c \, [e_m]^d \, R^0{}_{0cd} + [e_t]_a \, [e_l]^b \, [e_i]^c \, [e_m]^0 \, R^a{}_{bc0} \\
&\quad + [e_t]_a \, [e_l]^b \, [e_i]^0 \, [e_m]^d \, R^a{}_{b0d} + [e_t]_a \, [e_l]^0 \, [e_i]^c \, [e_m]^d \, R^a{}_{0cd} + [e_t]_0 \, [e_l]^b \, [e_i]^c \, [e_m]^d \, R^0{}_{bcd} \\
&\quad + [e_t]_a \, [e_l]^b \, [e_i]^c \, [e_m]^d \, R^a{}_{bcd} \\
&= [e_t]_0 \, [e_l]^b \, [e_i]^c \, [e_m]^0 \, R^0{}_{bc0} + [e_t]_0 \, [e_l]^b \, [e_i]^0 \, [e_m]^d \, R^0{}_{b0d} + [e_t]_0 \, [e_l]^b \, [e_i]^c \, [e_m]^d \, R^0{}_{bcd} \\
&\quad + [e_t]_a \, [e_l]^b \, [e_i]^c \, [e_m]^d \, R^a{}_{bcd} \\
&= \frac{1}{a^2} \{ \mathcal{H}' \delta_{bc} \delta^b_l \delta^c_i (V_m - B_m) - \mathcal{H}' \delta_{bd} \delta^b_l \delta^d_m (V_i - B_i) \\
&\qquad - [2\mathcal{H} \delta_{b[c} A_{,d]} - B_{b,[cd]} + \frac{1}{2} (B_{d,bc} - B_{c,bd}) - 2C'_{b[c,d]}] \delta^b_l \delta^d_m \delta^c_i \\
&\qquad + \mathcal{H}^2 \, (\delta^a_c \delta_{bd} - \delta^a_d \delta_{bc}) \, \delta^b_l \delta^d_m \delta^c_i (V_a - B_a) \} \\
&= \frac{1}{a^2} [\mathcal{H}' \delta_{li} (V_m - B_m) - \mathcal{H}' \delta_{lm} (V_i - B_i) - 2\mathcal{H} \delta_{l[i} A_{,m]} + B_{l,[im]} - \frac{1}{2} (B_{m,li} - B_{i,lm}) \\
&\qquad + 2C'_{l[i,m]} + \mathcal{H}^2 \delta_{lm} (V_i - B_i) - \mathcal{H}^2 \delta_{li} (V_m - B_m)] \\
&= \frac{1}{a^2} \left[-(\mathcal{H}^2 - \mathcal{H}') \delta_{li} (V_m - B_m) + (\mathcal{H}^2 - \mathcal{H}') \delta_{lm} (V_i - B_i) - 2\mathcal{H} \delta_{l[i} A_{,m]} \right. \\
&\qquad \left. + B_{[i,m]l} + 2C'_{l[i,m]} \right]
\end{aligned}
\tag{163}
$$

The $iljm$ component is

$$\begin{aligned}
R^F_{iljm} &= [e_i]_\mu [e_l]^\nu [e_j]^\kappa [e_m]^\lambda R^\mu{}_{\nu\kappa\lambda} \\
&= [e_i]_0 [e_l]^0 [e_j]^0 [e_m]^0 R^0{}_{000} + [e_i]_a [e_l]^0 [e_j]^0 [e_m]^0 R^a{}_{000} + [e_i]_0 [e_l]^b [e_j]^0 [e_m]^0 R^0{}_{b00} \\
&\quad + [e_i]_0 [e_l]^0 [e_j]^c [e_m]^0 R^0{}_{0c0} + [e_i]_0 [e_l]^0 [e_j]^0 [e_m]^d R^0{}_{00d} + [e_i]_a [e_l]^b [e_j]^0 [e_m]^0 R^a{}_{b00} \\
&\quad + [e_i]_a [e_l]^0 [e_j]^c [e_m]^0 R^a{}_{0c0} + [e_i]_a [e_l]^0 [e_j]^0 [e_m]^d R^a{}_{00d} + [e_i]_0 [e_l]^b [e_j]^c [e_m]^0 R^0{}_{bc0} \\
&\quad + [e_i]_0 [e_l]^b [e_j]^0 [e_m]^d R^0{}_{b0d} + [e_i]_0 [e_l]^0 [e_j]^c [e_m]^d R^0{}_{0cd} + [e_i]_a [e_l]^b [e_j]^c [e_m]^0 R^a{}_{bc0} \\
&\quad + [e_i]_a [e_l]^b [e_j]^0 [e_m]^d R^a{}_{b0d} + [e_i]_a [e_l]^0 [e_j]^c [e_m]^d R^a{}_{0cd} + [e_i]_0 [e_l]^b [e_j]^c [e_m]^d R^0{}_{bcd} \\
&\quad + [e_i]_a [e_l]^b [e_j]^c [e_m]^d R^a{}_{bcd} \\
&= [e_i]_a [e_l]^b [e_j]^c [e_m]^d R^a{}_{bcd} \\
&= \frac{1}{a^2}\{(\delta_{ia} + C_{ia} - \epsilon_{aih}\Omega^h)(\delta^b_l - C^b_l - \epsilon^b{}_{lh}\Omega^h)(\delta^c_j - C^c_j - \epsilon^c{}_{jh}\Omega^h)(\delta^d_m - C^d_m - \epsilon^d{}_{mh}\Omega^h) \\
&\qquad [\mathcal{H}^2\left(\delta^a_c\delta_{bd} - \delta^a_d\delta_{bc}\right)(1-2A) \\
&\qquad + \frac{1}{2}\mathcal{H}\left(\delta_{bd}\left(B_c{}^{,a} + B^a{}_{,c}\right) - \delta_{bc}\left(B_d{}^{,a} + B^a{}_{,d}\right) + 2\delta^a_c B_{(b,d)} - 2\delta^a_d B_{(b,c)}\right) \\
&\qquad + \mathcal{H}\left(\delta_{bd}C^{a\prime}_c - \delta_{bc}C^{a\prime}_d + \delta^a_c C'_{bd} - \delta^a_d C'_{bc} + 2\mathcal{H}\left(\delta^a_c C_{bd} - \delta^a_d C_{bc}\right)\right) \\
&\qquad + 2C^a_{(b,d),c} - 2C^a_{(b,c),d} + C_{bc}{}^{,a}{}_{,d} - C_{bd}{}^{,\alpha}{}_{,c}]\} \\
&= \frac{1}{a^2}\{\mathcal{H}^2(\delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj})(1-2A) \\
&\qquad + \frac{1}{2}\mathcal{H}(\delta_{lm}(B_{j,i} + B_{i,j}) - \delta_{lj}(B_{m,i} + B_{i,m}) + 2\delta_{ij}B_{(l,m)} - 2\delta_{im}B_{(l,j)}) \\
&\qquad + \mathcal{H}(\delta_{lm}C'_{ji} - \delta_{lj}C'_{im} + \delta_{ij}C'_{lm} - \delta_{im}C'_{lj} + 2\mathcal{H}(\delta_{ij}C_{lm} - \delta_{im}C_{lj})) \\
&\qquad + 2C_{i(l,m),j} - 2C_{i(l,j),m} + C_{lj,im} - C_{lm,ij} \\
&\qquad + \mathcal{H}^2(C_{ia} - \epsilon_{aih}\Omega^h)(\delta^a_j\delta_{lm} - \delta^a_m\delta_{lj}) + \mathcal{H}^2(-C^b_l - \epsilon^b{}_{lh}\Omega^h)(\delta_{ij}\delta_{bm} - \delta_{im}\delta_{bj}) \\
&\qquad + \mathcal{H}^2(-C^c_j - \epsilon^c{}_{jh}\Omega^h)(\delta_{ic}\delta_{lm} - \delta_{im}\delta_{lc}) + \mathcal{H}^2(-C^d_m - \epsilon^d{}_{mh}\Omega^h)(\delta_{ij}\delta_{ld} - \delta_{id}\delta_{lj})\} \\
&= \frac{1}{a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A) + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) \\
&\qquad + 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}]
\end{aligned} \tag{164}$$

We have written down the Riemann tensor in FNC and we are now prepared to write down the metric of FNC $g^F_{\mu\nu}$ under the background of the perturbed Robertson-Walker space-time. The metric has the form of

$$\begin{aligned}
g^F_{00}(Q) &= \eta_{00} - R^F_{0l0m}x^l_F x^m_F, \\
g^F_{0a}(Q) &= \eta_{0a} - \frac{2}{3}R^F_{0lam}x^l_F x^m_F, \\
g^F_{ab}(Q) &= \eta_{ab} - \frac{1}{3}R^F_{albm}x^l_F x^m_F.
\end{aligned} \tag{165}$$

Therefore, the metric is

$$g^F_{00}(Q) = \eta_{00} - \frac{1}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A) + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm}]x^l_F x^m_F \tag{166}$$

$$\begin{aligned}
g^F_{0a}(Q) = \eta_{0a} - \frac{2}{3a^2}[&-(\mathcal{H}^2 - \mathcal{H}')\delta_{li}(V_m - B_m) + (\mathcal{H}^2 - \mathcal{H}')\delta_{lm}(V_i - B_i) - 2\mathcal{H}\delta_{l[i}A_{,m]} \\
&+ B_{[i,m]l} + 2C'_{l[i,m]}]x^l_F x^m_F
\end{aligned} \tag{167}$$

$$g^F_{ab}(Q) = \eta_{ab} - \frac{1}{3a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A) + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) + 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}]x^l_F x^m_F \tag{168}$$

So far, we have finally got the specific form of the metric in FNC when the background space-time is the perturbed Robertdon-Walker space-time. We applied the FNC to the research of cosmology. We find that for a comoving observer on a time-like geodesic, we are able to find a locally inertial coordinate in which that in the neighborhood of the comoving observer the metric is the Minkowski metric plus some deviations.

8.4 The limitation of FNC

We have mentioned that the FNC can only be applied in the neighborhood of the comoving observer. However, how big this patch in which the metric is close to the Minkowski metric is not discussed carefully. For this purpose, we first ignore the perturbations since they are defined to be small. What we get is then the FNC metric under the Robertson-Walker space-time background.

$$\begin{aligned} g^F_{00}(Q) &= \eta_{00} - (H^2 - \frac{\ddot{a}}{a})\delta_{lm}x^l_F x^m_F \\ g^F_{0a}(Q) &= \eta_{0a} \\ g^F_{ab}(Q) &= \eta_{ab} - \frac{2}{3}H^2\delta_{i[j}\delta_{m]l}x^l_F x^m_F \end{aligned} \tag{169}$$

From Eq.(169) we find the condition that guarantees the metric to be close to Minkowski is

$$H^2x_F^2 < 1 \ \rightarrow \ x_F < H^{-1} \tag{170}$$

Thus, the patch that can be studied by FNC is much smaller than the Hubble horizon H^{-1}. This is the limitation of applying FNC to the study of cosmology. This is also the motivation for the construction of the conformal Fermi coordinates. In Part V we will discuss the CFC in detail and see how the CFC solves the problem that exist in FNC.[40]

Part V
Conformal Fermi Coordinates

In Part IV we have constructed the FNC and we know that the basic formalism and analytical formula hold in any arbitrary space-time meaning that we can apply any global metric to NFC. However, the drawback of FNC is that when we want to apply it to cosmology we use Robertson Walker metric as the global metric. However, Robertson-Walker metric brings H^2 to the Riemann tensor which we have seen in Eq.(96). This brings a problem to the size of the patch that can be covered by FNC because for $x_F > H^{-1}$ the deviation of the FNC metric from Minkowski space is too large. Thus the scale that can be studied is limited.

In order to solve this problem we use a 'trick' to take the scale factor out. The general idea is that we consider a conformal global metric and define the CFC coordinate based on this conformal metric. The CFC is first introduced in [41]. The Conformal Fermi coordinates are coordinates of a local observer that describes the space-time in a neighborhood of her worldline as a Robertson-Walker space-time. The corrections from the unperturbed Robertson-Walker space-time grow quadratically in the distance from the worldline when we apply perturbations to the metric. Depending on the structure of the space-time, the corrections can stay small on scales much larger than the Hubble horizon. Thus, these coordinates share all the advantages of FNC but are valid on super-horizon scales.

The main idea is very similar to the construction of the NFC. But we define the conformal metric $\tilde{g}_{\mu\nu}(x)$ and set up the orthonormal tetrads $[\tilde{e}_\alpha]^\mu$ and the coordinates x_F^i basing on the conformal metric $\tilde{g}_{\mu\nu}(x)$. We will now construct CFC in detail.

9 Basic Formalism and Analytical Formula

We would like to construct the conformal Fermi coordinates for a more general case. At this point we will not apply the Robertson-Walker metric to the back ground. Instead we will define a new CFC scale factor a_F which is not yet related to the scale factor a in the Robertson-Walker metric. Therefore, we will consider an arbitrary 'global metric' $g_{\mu\nu}(x)$ and the 'global coordinate' $x^\mu = \{t, x^i\}$. The same as what we have done in the construction of NFC we consider a free fall observer which determines a time-like central geodesic $x^\mu(\tau)$.

Now we want to construct a conformal space-time. We consider a space-time scalar a_F. And define a 'conformal proper time' η_F through

$$d\eta_F = a_F^{-1}(\tau)d\tau \tag{171}$$

Where the scale factor $a_F(\tau)$ is parameterized by the proper time τ along the time-like central geodesic. By integrating this differential equation we get the relation between the 'conformal proper time' η_F and the 'proper time' τ. So we can also write the scale factor as $a_F(\eta_F)$. In NFC we chose the proper time to be the time coordinate, in CFC instead we choose the conformal proper time η_F to be the time coordinate.

Now we need to define the slices of constant η_F. By using the CFC scale factor a_F we define a 'conformal metric' $\tilde{g}_{\mu\nu}(x)$ to take the scale factor out of the global metric around the time-like central geodesic.

$$\tilde{g}_{\mu\nu}(x) \equiv a_F^{-2}(\tau)g_{\mu\nu}(x) \tag{172}$$

Consider a spatial hypersurface passing through a point $P(\tau)$ on the time-like central geodesic. The infinitesimal space-time interval between two events from a point $P(\tau)$ to a point $Q(\tau; x^i)$ on the space-like geodesic is

$$ds^2 = a_F^2(\tau)\tilde{g}_{ab}(x)dx^a dx^b = g_{ab}(x)dx^a dx^b \tag{173}$$

While the conformal infinitesimal space-time interval between two events from a point $P(\tau)$ to a point $Q(\tau; x^i)$ on the space-like geodesic is

$$d\tilde{s}^2 = \tilde{g}_{ab}(x)dx^a dx^b \tag{174}$$

We define the four tetrads as in the construction of FNC. But this time we define the four tetrads $[\tilde{e}_\alpha]^\mu$ also in the conformal space-time. Which implies

$$[\tilde{e}_\alpha]^\mu = a_F(\eta_F)[e_\alpha]^\mu \tag{175}$$

The tangent vector $\left.\frac{\partial x^\mu(s)}{\partial \tilde{s}}\right|_0$ can then be written by a composition of the spatial tetrads which is a set of the local coordinate bases.

$$\left.\frac{\partial x^\mu(s)}{\partial \tilde{s}}\right|_0 = \beta^a[\tilde{e}_a]^\mu \tag{176}$$

Where $d\tilde{s} = \sqrt{\tilde{g}_{ab}(x)dx^a dx^b}$.

As what has been done for the FNC we expand the global coordinate $x^\mu(\tilde{s}_Q)$ of a point Q on a space-like geodesic through point P around point P.

$$x^\mu(\tilde{s}_Q) = x^\mu(0) + \tilde{s}_Q \frac{\partial x^\mu(\tilde{s})}{\partial \tilde{s}}\bigg|_0 + \frac{1}{2}\tilde{s}_Q^2 \frac{\partial^2 x^\mu(\tilde{s})}{\partial \tilde{s}^2}\bigg|_0 + \frac{1}{6}\tilde{s}_Q^3 \frac{\partial^3 x^\mu(\tilde{s})}{\partial \tilde{s}^3}\bigg|_0 \tag{177}$$

Define CFC coordinate x_F^i base on conformal metric $\tilde{g}_{ab}$.

$$x_F^i = \beta^i \tilde{s}_Q \tag{178}$$

We can see clearly that this definition makes sense by considering $\beta^i = (1, 0, 0)$. This special case just gives $x_F^i = (\tilde{s}_Q, 0, 0)$. Where the conformal proper space-time interval between event P and event Q is

$$\tilde{s}_Q = \int \sqrt{\tilde{g}_{ab}(x) dx^a dx^b} \tag{179}$$

Now with these definitions, we are able to rewrite the global coordinate $x^\mu(\tilde{s}_Q)$ in terms of the CFC coordinate.

$$\begin{aligned}
s_Q \frac{\partial x^\mu(s)}{\partial s}\bigg|_0 &= a_F^{-1}(\tau) s_Q \frac{\partial x^\mu(s)}{\partial \tilde{s}}\bigg|_0 = \tilde{s}_Q \beta^i [\tilde{e}_i]^\mu = a_F(\tau) x_F^i [e_i]^\mu \\
\frac{1}{2} s_Q^2 \frac{\partial^2 x^\mu(s)}{\partial s^2}\bigg|_0 &= \frac{1}{2} s_Q^2 a_F^{-2}(\tau) \frac{\partial^2 x^\mu(s)}{\partial \tilde{s}^2}\bigg|_0 = -\frac{1}{2}\tilde{s}_Q^2 \tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}}\bigg|_P \\
&= -\frac{1}{2}\tilde{s}_Q^2 \tilde{\Gamma}^\mu_{\alpha\beta} \beta^i [\tilde{e}_i]^\alpha \beta^j [\tilde{e}_j]^\beta = -\frac{1}{2} a_F^2(\tau) \tilde{\Gamma}^\mu_{\alpha\beta} [e_i]^\alpha [e_j]^\beta x_F^i x_F^j \\
\frac{1}{6} s_Q^3 \frac{\partial^3 x^\mu(s)}{\partial s^3}\bigg|_0 &= \frac{1}{6}\tilde{s}_Q^3 \frac{\partial^3 x^\mu(s)}{\partial \tilde{s}^3}\bigg|_0 = -\frac{1}{6}\tilde{s}_Q^3 \frac{d}{d\tilde{s}}\left(\tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tilde{s}} \frac{dx^\beta}{d\tilde{s}}\right)_P \\
&= -\frac{1}{6}\tilde{s}_Q^3 \left(\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_P \beta^i [\tilde{e}_i]^\alpha \beta^j [\tilde{e}_j]^\beta \beta^k [\tilde{e}_k]^\gamma \\
&= -\frac{1}{6} a_F^3(\tau) \left(\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_P [e_i]^\alpha [e_j]^\beta [e_k]^\gamma x_F^i x_F^j x_F^k
\end{aligned} \tag{180}$$

Where

$$[\tilde{e}_a]^\mu = a_F(\tau)[e_a]^\mu \tag{181}$$

Thus $x^\mu(Q)$ can be written in terms of CFC coordinate

$$\begin{aligned}
x_Q^\mu =& P + a_F(\tau) [e_i]_P^\mu x_F^i - \frac{1}{2} a_F^2(\tau) \tilde{\Gamma}^\mu_{\alpha\beta}\bigg|_P [e_i]_P^\alpha [e_j]_P^\beta x_F^i x_F^j \\
&- \frac{1}{6} a_F^3(\tau) \left[\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right]_P [e_i]_P^\alpha [e_j]_P^\beta [e_k]_P^\gamma x_F^i x_F^j x_F^k
\end{aligned} \tag{182}$$

Where the Christoffel symbol with a tilde is based on conformal metric $\tilde{g}_{\mu\nu}(x) = a_F^{-2}(x) g_{\mu\nu}(x)$. The metric should satisfied $\tilde{g}_{\mu\nu}\tilde{g}^{\mu\nu} = \delta^\mu_\mu$. To raise the index we simply get

$$\tilde{g}^{\mu\nu}(x) = a_F^2(x) g^{\mu\nu}(x) \tag{183}$$

Use $\tilde{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\tilde{g}^{\mu\sigma}(\tilde{g}_{\nu\sigma,\rho} + \tilde{g}_{\rho\sigma,\nu} - \tilde{g}_{\nu\rho,\sigma})$ we get

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\alpha\beta} &= \Gamma^\mu_{\alpha\beta} - a_F^{-1}\nabla_\beta a_F \delta^\mu_\alpha - a_F^{-1}\nabla_\alpha a_F \delta^\mu_\beta + a_F^{-1}\nabla_\lambda a_F g^{\mu\sigma} g_{\alpha\beta} \\
&= \Gamma^\mu_{\alpha\beta} - \delta^\mu_\alpha \nabla_\beta \ln a_F - \delta^\mu_\beta \nabla_\alpha \ln a_F + g_{\alpha\beta} g^{\mu\lambda} \nabla_\lambda \ln a_F \\
&= \Gamma^\mu_{\alpha\beta} + C^\mu_{\alpha\beta}
\end{aligned} \tag{184}$$

Where $C^{\mu}_{\alpha\beta} := -\delta^{\mu}_{\alpha}\nabla_{\beta}\ln a_F - \delta^{\mu}_{\beta}\nabla_{\alpha}\ln a_F + g_{\alpha\beta}g^{\mu\lambda}\nabla_{\lambda}\ln a_F$ The gradient of $a_F(\tau)$ along the central geodesic is along the time direction

$$\frac{D\ln a_F}{Dx^{\mu}} = -\left(\ln a_F\right)'\left[\tilde{e}_0\right]_{\mu} \tag{185}$$

The second derivatives

$$\begin{aligned}\nabla_{\alpha}\nabla_{\beta}\ln a_F &= \nabla_{\alpha}\left[-\left(\ln a_F\right)'\left[\tilde{e}_0\right]_{\beta}\right]\\ &= -\frac{D\left[\left(\ln a_F\right)'\left[\tilde{e}_0\right]_{\beta}\right]}{Dx^{\alpha}}\\ &= \left(\ln a_F\right)''\left[\tilde{e}_0\right]_{\beta}\left[\tilde{e}_0\right]_{\alpha}\end{aligned} \tag{186}$$

Where we have used the fact that the covariant derivative just reduces to the ordinary derivative when acting on a scalar field. With the first and second derivatives, we are able to compute the Riemann Tensor.

$$\begin{aligned}\tilde{R}^{\mu}_{\nu\rho\sigma} &:= \tilde{\Gamma}^{\mu}_{\nu\sigma,\rho} - \tilde{\Gamma}^{\mu}_{\nu\rho,\sigma} + \tilde{\Gamma}^{\epsilon}_{\nu\sigma}\tilde{\Gamma}^{\mu}_{\rho\epsilon} - \tilde{\Gamma}^{\epsilon}_{\nu\rho}\tilde{\Gamma}^{\mu}_{\sigma\epsilon}\\ &= \Gamma^{\mu}_{\nu\sigma,\rho} + C^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} - C^{\mu}_{\nu\rho,\sigma} + (\Gamma^{\epsilon}_{\nu\sigma} + C^{\epsilon}_{\nu\sigma})(\Gamma^{\mu}_{\rho\epsilon} + C^{\mu}_{\rho\epsilon}) - (\Gamma^{\epsilon}_{\nu\rho} + C^{\epsilon}_{\nu\rho})(\Gamma^{\mu}_{\sigma\epsilon} + C^{\mu}_{\sigma\epsilon})\\ &= R^{\mu}_{\nu\rho\sigma} + C^{\mu}_{\nu\sigma,\rho} - C^{\mu}_{\nu\rho,\sigma} + \Gamma^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon} + \Gamma^{\mu}_{\rho\epsilon}C^{\epsilon}_{\nu\sigma} + C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon} - \Gamma^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon} - \Gamma^{\mu}_{\sigma\epsilon}C^{\epsilon}_{\nu\rho} - C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon}\\ &= R^{\mu}_{\nu\rho\sigma} + C^{\mu}_{\nu\sigma;\rho} - C^{\mu}_{\nu\rho;\sigma} - C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon} + C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon}\end{aligned} \tag{187}$$

Where we have used

$$C^{\mu}_{\nu\sigma;\rho} = C^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\rho\epsilon}C^{\epsilon}_{\nu\sigma} - \Gamma^{\epsilon}_{\rho\nu}C^{\mu}_{\epsilon\sigma} - \Gamma^{\epsilon}_{\rho\sigma}C^{\mu}_{\epsilon\nu} \tag{188}$$

$$-C^{\mu}_{\nu\rho;\sigma} = -C^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\sigma\epsilon}C^{\epsilon}_{\nu\rho} + \Gamma^{\epsilon}_{\sigma\nu}C^{\mu}_{\epsilon\rho} + \Gamma^{\epsilon}_{\rho\sigma}C^{\mu}_{\epsilon\nu} \tag{189}$$

Where

$$\begin{aligned}C^{\mu}_{\nu\sigma;\rho} &= \frac{D}{Dx^{\rho}}[-\delta^{\mu}_{\nu}\nabla_{\sigma}\ln a_F - \delta^{\mu}_{\sigma}\nabla_{\nu}\ln a_F + g_{\nu\sigma}g^{\mu\lambda}\nabla_{\lambda}\ln a_F]\\ &= -\delta^{\mu}_{\nu}\nabla_{\rho}\nabla_{\sigma}\ln a_F - \delta^{\mu}_{\sigma}\nabla_{\rho}\nabla_{\nu}\ln a_F + g_{\nu\sigma}g^{\mu\lambda}\nabla_{\rho}\nabla_{\lambda}\ln a_F\\ &\quad + g_{\nu\sigma;\rho}g^{\mu\lambda}\nabla_{\lambda}\ln a_F + g_{\nu\sigma}g^{\mu\lambda}{}_{;\rho}\nabla_{\lambda}\ln a_F\\ &= -\delta^{\mu}_{\nu}\nabla_{\rho}\nabla_{\sigma}\ln a_F - \delta^{\mu}_{\sigma}\nabla_{\rho}\nabla_{\nu}\ln a_F + g_{\nu\sigma}g^{\mu\lambda}\nabla_{\rho}\nabla_{\lambda}\ln a_F\end{aligned} \tag{190}$$

$$\begin{aligned}C^{\mu}_{\nu\rho;\sigma} &= \frac{D}{Dx^{\sigma}}[-\delta^{\mu}_{\nu}\nabla_{\rho}\ln a_F - \delta^{\mu}_{\rho}\nabla_{\nu}\ln a_F + g_{\nu\rho}g^{\mu\lambda}\nabla_{\lambda}\ln a_F]\\ &= -\delta^{\mu}_{\nu}\nabla_{\sigma}\nabla_{\rho}\ln a_F - \delta^{\mu}_{\rho}\nabla_{\sigma}\nabla_{\nu}\ln a_F + g_{\nu\rho}g^{\mu\lambda}\nabla_{\sigma}\nabla_{\lambda}\ln a_F\\ &\quad + g_{\nu\rho;\sigma}g^{\mu\lambda}\nabla_{\lambda}\ln a_F + g_{\nu\rho}g^{\mu\lambda}{}_{;\sigma}\nabla_{\lambda}\ln a_F\\ &= -\delta^{\mu}_{\nu}\nabla_{\sigma}\nabla_{\rho}\ln a_F - \delta^{\mu}_{\rho}\nabla_{\sigma}\nabla_{\nu}\ln a_F + g_{\nu\rho}g^{\mu\lambda}\nabla_{\sigma}\nabla_{\lambda}\ln a_F\end{aligned} \tag{191}$$

$$\begin{aligned}C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon} &= (-\delta^{\epsilon}_{\nu}\nabla_{\rho}\ln a_F - \delta^{\epsilon}_{\rho}\nabla_{\nu}\ln a_F + g_{\nu\rho}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F)\\ &\quad (-\delta^{\mu}_{\sigma}\nabla_{\epsilon}\ln a_F - \delta^{\mu}_{\epsilon}\nabla_{\sigma}\ln a_F + g_{\sigma\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F)\\ &= 2\nabla_{\rho}\ln a_F\delta^{\mu}_{\sigma}\nabla_{\nu}\ln a_F + \delta^{\mu}_{\nu}\nabla_{\rho}\ln a_F\nabla_{\sigma}\ln a_F + \delta^{\mu}_{\rho}\nabla_{\nu}\ln a_F\nabla_{\sigma}\ln a_F\\ &\quad - g_{\nu\rho}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F\delta^{\mu}_{\sigma}\nabla_{\epsilon}\ln a_F - g_{\nu\rho}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F\delta^{\mu}_{\epsilon}\nabla_{\sigma}\ln a_F\\ &\quad - g_{\sigma\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F\delta^{\epsilon}_{\nu}\nabla_{\rho}\ln a_F - g_{\sigma\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F\delta^{\epsilon}_{\rho}\nabla_{\nu}\ln a_F\\ &\quad + g_{\nu\rho}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F g_{\sigma\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F\end{aligned} \tag{192}$$

$$
\begin{aligned}
C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon} = & 2\nabla_{\sigma}\ln a_F \delta^{\mu}_{\rho}\nabla_{\nu}\ln a_F + \delta^{\mu}_{\nu}\nabla_{\sigma}\ln a_F\nabla_{\rho}\ln a_F + \delta^{\mu}_{\sigma}\nabla_{\nu}\ln a_F\nabla_{\rho}\ln a_F \\
& - g_{\nu\sigma}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F\delta^{\mu}_{\rho}\nabla_{\epsilon}\ln a_F - g_{\nu\sigma}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F\delta^{\mu}_{\epsilon}\nabla_{\rho}\ln a_F \\
& - g_{\rho\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F\delta^{\epsilon}_{\nu}\nabla_{\sigma}\ln a_F - g_{\rho\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F\delta^{\epsilon}_{\sigma}\nabla_{\nu}\ln a_F \\
& + g_{\nu\sigma}g^{\epsilon\lambda}\nabla_{\lambda}\ln a_F g_{\rho\epsilon}g^{\mu\omega}\nabla_{\omega}\ln a_F
\end{aligned} \tag{193}
$$

To compute the CFC metric, we need to compute the derivatives of the coordinate transformation

$$
\begin{aligned}
\frac{\partial x^{\mu}}{\partial x^{0}_{F}}(Q) = & \frac{\partial x^{\mu}}{\partial \eta_F}(P) + \frac{\partial}{\partial \eta_F}\left[a_F(\tau)\left[e_i\right]^{\mu}_{P}\right]x^i_F - \frac{1}{2}\frac{\partial}{\partial \eta_F}\left[a_F(\tau)^2\tilde{\Gamma}^{\mu}_{\alpha\beta}\Big|_P\left[e_i\right]^{\alpha}_P\left[e_j\right]^{\beta}_P\right]x^i_F x^j_F + \mathcal{O}\left(x^3_F\right) \\
= & a_F(\tau)[e_0]^{\mu} + a'_F(\tau)[e_i]^{\mu}x^i_F - a_F(\tau)^2\ \Gamma^{\mu}_{\alpha\beta}\Big|_P\left[e_0\right]^{\alpha}\left[e_i\right]^{\beta}x^i_F \\
& - a_F(\tau)a'_F(\tau)\ \tilde{\Gamma}^{\mu}_{\alpha\beta}\Big|_P\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}x^i_F x^j_F - \frac{1}{2}a_F(\tau)^3\ \tilde{\Gamma}^{\mu}_{\alpha\beta,\gamma}\Big|_P\left[e_0\right]^{\gamma}\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}x^i_F x^j_F \\
& + a_F(\tau)^3\ \tilde{\Gamma}^{\mu}_{\alpha\sigma}\Big|_P\ \Gamma^{\sigma}_{\beta\gamma}\big|_P\left[e_0\right]^{\gamma}\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}x^i_F x^j_F + \mathcal{O}\left(x^3_F\right) \\
= & a_F(\tau)[e_0]^{\mu} + \left[a'_F(\tau)[e_i]^{\mu} - a^2_F(\tau)\ \Gamma^{\mu}_{\alpha\beta}\Big|_P\left[e_0\right]^{\alpha}\left[e_i\right]^{\beta}\right]x^i_F \\
& - \left[a_F(\tau)a'_F(\tau)\ \tilde{\Gamma}^{\mu}_{\alpha\beta}\Big|_P\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}\right. \\
& \left. + \left[\frac{1}{2}a^3_F(\tau)\ \tilde{\Gamma}^{\mu}_{\alpha\beta,\gamma}\Big|_P - a^3_F(\tau)\ \tilde{\Gamma}^{\mu}_{\alpha\sigma}\Big|_P\ \Gamma^{\sigma}_{\beta\gamma}\big|_P\right]\left[e_0\right]^{\gamma}\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}\right]x^i_F x^j_F + \mathcal{O}\left(x^3_F\right)
\end{aligned} \tag{194}
$$

$$
\begin{aligned}
\frac{\partial x^{\mu}}{\partial x^{l}_{F}}(Q) = & 0 + a_F(\tau)[e_i]^{\mu}\delta^i_l - \frac{1}{2}a_F(\tau)^2\tilde{\Gamma}^{\mu}_{\alpha\beta}[e_i]^{\alpha}[e_j]^{\beta}\left[\delta^i_l x^j_F + \delta^j_l x^i_F\right] \\
& - \frac{1}{6}a^3_F(\tau)\left[\tilde{\Gamma}^{\mu}_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^{\mu}_{\sigma\alpha}\tilde{\Gamma}^{\sigma}_{\beta\gamma}\right]_P\left[e_i\right]^{\alpha}\left[e_j\right]^{\beta}\left[e_k\right]^{\gamma}\left(\delta^i_l x^j_F x^k_F + \delta^j_l x^i_F x^k_F + \delta^k_l x^j_F x^i_F\right) \\
= & a_F(\tau)[e_l]^{\mu} - a_F(\tau)^2\tilde{\Gamma}^{\mu}_{\alpha\beta}[e_l]^{\alpha}[e_i]^{\beta}x^i_F \\
& - \frac{1}{6}a^3_F(\tau)\left[\tilde{\Gamma}^{\mu}_{\alpha\beta,\gamma} + 2\tilde{\Gamma}^{\mu}_{\beta\gamma,\alpha} - 2\tilde{\Gamma}^{\mu}_{\gamma\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta} - 4\tilde{\Gamma}^{\mu}_{\alpha\sigma}\tilde{\Gamma}^{\sigma}_{\beta\gamma}\right][e_l]^{\alpha}[e_j]^{\beta}[e_k]^{\gamma}x^j_F x^k_F
\end{aligned} \tag{195}
$$

Then we can find the metric of CFC order-by-order in x^i_F using

$$
g^F_{\mu\nu}\left(x_F\right) = \frac{\partial x^{\alpha}}{\partial x^{\mu}_F}\frac{\partial x^{\beta}}{\partial x^{\nu}_F}g_{\alpha\beta}(x) \tag{196}
$$

However, it is quite lengthy to go through this calculation under the global metric $g_{\alpha\beta}$. So we try to project various geometric quantities into the 'conformal metric' $\tilde{g}_{\alpha\beta}$. We first find that

$$
g^F_{\mu\nu}\left(x_F\right) = \frac{\partial x^{\alpha}}{\partial x^{\mu}_F}\frac{\partial x^{\beta}}{\partial x^{\nu}_F}g_{\alpha\beta}(x) = \frac{\partial x^{\alpha}}{\partial x^{\mu}_F}\frac{\partial x^{\beta}}{\partial x^{\nu}_F}a^2_F(\tau)\tilde{g}_{\alpha\beta}(x) \tag{197}
$$

We will calculate the metric of CFC for each component. First, we expand the conformal metric around event P.

$$
\begin{aligned}
\tilde{g}_{\alpha\beta}(Q) = & \tilde{g}_{\alpha\beta}\big|_P + \tilde{g}_{\alpha\beta,\mu}\big|_P\left[x^{\mu}_Q - x^{\mu}_P\right] + \frac{1}{2}\tilde{g}_{\alpha\beta,\mu\nu}\Big|_P[x^{\mu}_Q - x^{\mu}_P][x^{\nu}_Q - x^{\nu}_P] \\
= & \tilde{g}_{\alpha\beta}\big|_P + \tilde{g}_{\alpha\beta,\mu}\big|_P\left[\tilde{e}_i\right]^{\mu}_P x^i_F + \frac{1}{2}\left[\tilde{g}_{\alpha\beta,\mu\nu} - \tilde{g}_{\alpha\beta,\sigma}\tilde{\Gamma}^{\sigma}_{\mu\nu}\right]_P\left[\tilde{e}_i\right]^{\mu}_P\left[\tilde{e}_j\right]^{\nu}_P x^i_F x^j_F
\end{aligned} \tag{198}
$$

Where we have used the $x^\mu(Q)$ in terms of CFC coordinate.

$$\begin{aligned} x_Q^\mu =& P + [\tilde{e}_i]_P^\mu \, x_F^i - \frac{1}{2}\tilde{\Gamma}^\mu_{\alpha\beta}\Big|_P [\tilde{e}_i]_P^\alpha \, [\tilde{e}_j]_P^\beta \, x_F^i x_F^j \\ &- \frac{1}{6}\left[\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right]_P [\tilde{e}_i]_P^\alpha \, [\tilde{e}_j]_P^\beta \, [\tilde{e}_k]_P^\gamma \, x_F^i x_F^j x_F^k \end{aligned} \tag{199}$$

And the derivatives of the coordinate transformation

$$\begin{aligned} \frac{\partial x^\mu}{\partial x_F^0}(Q) &= \frac{\partial x^\mu}{\partial \eta_F}(P) + \frac{\partial}{\partial \eta_F}[\tilde{e}_i]_P^\mu \, x_F^i - \frac{1}{2}\frac{\partial}{\partial \eta_F}\left[\tilde{\Gamma}^\mu_{\alpha\beta}\Big|_P [\tilde{e}_i]_P^\alpha \, [\tilde{e}_j]_P^\beta\right] x_F^i x_F^j + \mathcal{O}\left(x_F^3\right) \\ &= [\tilde{e}_t]_P^\mu - \tilde{\Gamma}^\mu_{\alpha\beta}\Big|_P [\tilde{e}_i]_P^\alpha \, [\tilde{e}_t]_P^\beta \, x_F^i - \frac{1}{2}\left[\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} - 2\tilde{\Gamma}^\mu_{\sigma\beta}\tilde{\Gamma}^\sigma_{\gamma\alpha}\right]_P [\tilde{e}_t]_P^\gamma \, [\tilde{e}_i]_P^\alpha \, [\tilde{e}_j]_P^\beta \, x_F^i x_F^j \end{aligned} \tag{200}$$

$$\begin{aligned} \frac{\partial x^\mu}{\partial x_F^l}(Q) &= [\tilde{e}_l]^\mu - \tilde{\Gamma}^\mu_{\alpha\beta}[\tilde{e}_l]^\alpha[\tilde{e}_i]^\beta x_F^i \\ &\quad - \frac{1}{6}\left[\tilde{\Gamma}^\mu_{\alpha\beta,\gamma} + 2\tilde{\Gamma}^\mu_{\beta\gamma,\alpha} - 2\tilde{\Gamma}^\mu_{\gamma\sigma}\tilde{\Gamma}^\sigma_{\alpha\beta} - 4\tilde{\Gamma}^\mu_{\alpha\sigma}\tilde{\Gamma}^\sigma_{\beta\gamma}\right][\tilde{e}_l]^\alpha[\tilde{e}_j]^\beta[\tilde{e}_k]^\gamma x_F^j x_F^k \end{aligned} \tag{201}$$

For the 00 component $g_{00}^F(Q) = \frac{\partial x^\alpha}{\partial x_F^0}\frac{\partial x^\beta}{\partial x_F^0}a_F^2(\tau)\tilde{g}_{\alpha\beta}(Q)$.
The zeroth order term is

$$a_F^2(\tau)[\tilde{e}_0]^\alpha[\tilde{e}_0]^\beta\tilde{g}_{\alpha\beta}(P) = -a_F^2(\tau) \tag{202}$$

The first order term is

$$a_F^2(\eta_F)\left(\tilde{g}_{\alpha\beta,\rho} - \tilde{g}_{\sigma\beta}\tilde{\Gamma}^\sigma_{\rho\alpha} - \tilde{g}_{\alpha\sigma}\tilde{\Gamma}^\sigma_{\rho\beta}\right)_P [\tilde{e}_0]_P^\alpha \, [\tilde{e}_0]_P^\beta \, [\tilde{e}_i]_P^\rho \, x_F^i \quad = \tilde{g}_{\alpha\beta;\mu}(P)[\tilde{e}_0]^\alpha[\tilde{e}_0]^\beta[\tilde{e}_i]^\mu = 0 \tag{203}$$

The second order term is

$$\begin{aligned} & a_F^2(\eta_F)\left(\frac{1}{2}\tilde{g}_{\mu\nu,\alpha\beta} - \frac{1}{2}\tilde{g}_{\mu\nu,\sigma}\tilde{\Gamma}^\sigma_{\alpha\beta} - 2\tilde{g}_{\mu\gamma,\alpha}\tilde{\Gamma}^\gamma_{\beta\nu} - \tilde{g}_{\mu\gamma}\tilde{\Gamma}^\gamma_{\alpha\beta,\nu} + 2\tilde{g}_{\mu\gamma}\tilde{\Gamma}^\gamma_{\sigma\beta}\tilde{\Gamma}^\sigma_{\nu\alpha} + \tilde{g}_{\gamma\sigma}\tilde{\Gamma}^\gamma_{\alpha\mu}\tilde{\Gamma}^\sigma_{\beta\nu}\right)_P \\ & [\tilde{e}_0]_P^\mu \, [\tilde{e}_0]_P^\nu \, [\tilde{e}_l]_P^\alpha \, [\tilde{e}_m]_P^\beta \, x_F^l x_F^m \\ & = -a_F^2(\eta_F)\tilde{R}^F_{0\ell 0m}x_F^l x_F^m \end{aligned} \tag{204}$$

For the $0a$ component $g_{0a}^F(Q) = \frac{\partial x^\alpha}{\partial x_F^0}\frac{\partial x^\beta}{\partial x_F^a}a_F^2(\tau)\tilde{g}_{\alpha\beta}(Q)$.
The zeroth order term is

$$a_F^2(\eta_F)\,[\tilde{e}_0]_P^\alpha\,[\tilde{e}_a]_P^\beta\,\tilde{g}_{\alpha\beta} = 0 \tag{205}$$

The first order term is

$$a_F^2(\eta_F)\left(\tilde{g}_{\alpha\beta,\rho} - \tilde{g}_{\sigma\beta}\tilde{\Gamma}^\sigma_{\rho\alpha} - \tilde{g}_{\alpha\sigma}\tilde{\Gamma}^\sigma_{\rho\beta}\right)_P [\tilde{e}_0]_P^\alpha \, [\tilde{e}_a]_P^\beta \, [\tilde{e}_i]_P^\rho \, x_F^i \quad = \tilde{g}_{\alpha\beta;\mu}(P)[\tilde{e}_0]^\alpha[\tilde{e}_a]^\beta[\tilde{e}_i]^\mu = 0 \tag{206}$$

The second order term is

$$\begin{aligned} & a_F^2(\eta_F)\left[\frac{1}{2}\tilde{g}_{\mu\nu,\alpha\beta} - \frac{1}{2}\tilde{g}_{\mu\nu,\sigma}\tilde{\Gamma}^\sigma_{\alpha\beta} - \tilde{g}_{\mu\gamma,\alpha}\tilde{\Gamma}^\gamma_{\beta\nu} - \tilde{g}_{\nu\gamma,\alpha}\tilde{\Gamma}^\gamma_{\beta\mu} - \frac{1}{2}\tilde{g}_{\nu\gamma}\tilde{\Gamma}^\gamma_{\alpha\beta,\mu} + \tilde{g}_{\nu\gamma}\tilde{\Gamma}^\gamma_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\mu\beta} + \tilde{g}_{\gamma\sigma}\tilde{\Gamma}^\gamma_{\alpha\mu}\tilde{\Gamma}^\sigma_{\beta\nu}\right. \\ & \left. -\frac{1}{6}\tilde{g}_{\mu\lambda}\left(\tilde{\Gamma}^\lambda_{\alpha\beta,\nu} + 2\tilde{\Gamma}^\lambda_{\nu\alpha,\beta} - 2\tilde{\Gamma}^\lambda_{\sigma\nu}\tilde{\Gamma}^\sigma_{\alpha\beta} - 4\tilde{\Gamma}^\lambda_{\sigma\beta}\tilde{\Gamma}^\sigma_{\alpha\nu}\right)\right]_P [\tilde{e}_0]_P^\mu \, [\tilde{e}_a]_P^\nu \, [\tilde{e}_l]_P^\alpha \, [\tilde{e}_m]_P^\beta \, x_F^l x_F^m \\ & = -a_F^2(\eta_F)\frac{2}{3}\tilde{R}^F_{0lam}x_F^l x_F^m \end{aligned} \tag{207}$$

For ab component $g_{ab}^F(Q) = \frac{\partial x^\alpha}{\partial x_F^a}\frac{\partial x^\beta}{\partial x_F^b}a_F^2(\tau)\tilde{g}_{\alpha\beta}(Q)$.
The zeroth order term is

$$a_F^2(\eta_F)\,[\tilde{e}_a]_P^\alpha\,[\tilde{e}_b]_P^\beta\,\tilde{g}_{\alpha\beta} = a_F^2(\eta_F)\delta_{ab} \tag{208}$$

The first order term is

$$a_F^2(\eta_F)\left(\tilde{g}_{\alpha\beta,\rho}-\tilde{g}_{\sigma\beta}\tilde{\Gamma}^{\sigma}_{\rho\alpha}-\tilde{g}_{\alpha\sigma}\tilde{\Gamma}^{\sigma}_{\rho\beta}\right)_P[\tilde{e}_a]^{\alpha}_P\,[\tilde{e}_b]^{\beta}_P\,[\tilde{e}_i]^{\rho}_P\,x^i_F\quad=\tilde{g}_{\alpha\beta;\mu}(P)[\tilde{e}_a]^{\alpha}[\tilde{e}_b]^{\beta}[\tilde{e}_i]^{\mu}=0 \tag{209}$$

The second order term is

$$\begin{aligned}&a_F^2(\eta_F)\left[\frac{1}{2}\tilde{g}_{\mu v,\alpha\beta}-\frac{1}{2}\tilde{g}_{\mu v,\sigma}\tilde{\Gamma}^{\sigma}_{\alpha\beta}-2\tilde{g}_{\mu\gamma,\alpha}\tilde{\Gamma}^{\gamma}_{v\beta}+\tilde{g}_{\gamma\sigma}\tilde{\Gamma}^{\gamma}_{\alpha\mu}\tilde{\Gamma}^{\sigma}_{\beta v}\right.\\&\left.-\frac{1}{3}\tilde{g}_{\mu\lambda}\left(\tilde{\Gamma}^{\lambda}_{\alpha\beta,v}+2\tilde{\Gamma}^{\lambda}_{v\alpha,\beta}-2\tilde{\Gamma}^{\lambda}_{\sigma v}\tilde{\Gamma}^{\sigma}_{\alpha\beta}-4\tilde{\Gamma}^{\lambda}_{\sigma\beta}\tilde{\Gamma}^{\sigma}_{\alpha v}\right)\right]_P[\tilde{e}_a]^{\mu}_P\,[\tilde{e}_b]^{v}_P\,[\tilde{e}_l]^{\alpha}_P\,[\tilde{e}_m]^{\beta}_P\,x^l_Fx^m_F\\&=-a_F^2(\eta_F)\frac{1}{3}\tilde{R}^F_{albm}x^l_Fx^m_F\end{aligned} \tag{210}$$

To summarize the calculations we have done and at the end we get the metric of CFC to the quadratic corrections.[40][42]

$$\begin{aligned}g^F_{00}(Q)&=a_F^2(\eta_F)\left[\eta_{00}-\tilde{R}^F_{0k0l}\Big|_P x^k_Fx^l_F\right],\\g^F_{0a}(Q)&=a_F^2(\eta_F)\left[\eta_{0a}-\frac{2}{3}\tilde{R}^F_{0kal}\Big|_P x^k_Fx^l_F\right],\\g^F_{ab}(Q)&=a_F^2(\eta_F)\left[\eta_{ab}-\frac{1}{3}\tilde{R}^F_{akbl}\Big|_P x^k_Fx^l_F\right].\end{aligned} \tag{211}$$

Similar to the FNC, we find that the metric in CFC is

$$g^F_{\mu\nu}=a_F^2(\eta_F)\left[\eta_{\mu\nu}+\mathcal{O}(\tilde{R}^F_{\mu l\nu m}x^l_Fx^m_F)\right]. \tag{212}$$

This is very similar to the FNC metric but the benefits of using $\tilde{R}_{\mu l\nu m}$ here are very significant. We have mentioned that the Robertson-Walker metric brings H^2 to the Riemann tensor, which brings a problem to the size of the patch that can be covered by FNC because for $x_F > H^{-1}$ the deviation of the FNC metric from Minkowski space is too large. This limits the scale that can be studied. In the next section, we will show that when considering the Robertson-Walker metric as the background metric the H^2 term can be canceled which solved the problem of the limited patch in the case of FNC.

10 Linear order calculations

In section 9, Conformal Fermi Coordinate is constructed with some random 'global metric' $g_{\mu\nu}(x)$. When studying cosmological problems we apply the Robertson-Walker metric to the background space-time. For the real universe we also consider the perturbations in the 'global metric'. The 'global metric' is given by

$$g_{00}:=-a^2(1+2A),\quad g_{0a}:=-a^2B_a,\quad g_{ab}:=a^2\left(\delta_{ab}+2C_{ab}\right) \tag{213}$$

Now we want to do the linear calculations of $\tilde{R}^F$ which is based on $\tilde{g}_{\mu\nu}=\frac{1}{a_F^2}g_{\mu\nu}$.

$$\begin{aligned}\tilde{R}^F_{\alpha k\beta l}&=\tilde{R}^{\mu}_{\nu\rho\sigma}[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma\\&=\left(R^{\mu}_{\nu\rho\sigma}+C^{\mu}_{\nu\sigma;\rho}-C^{\mu}_{\nu\rho;\sigma}-C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon}+C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon}\right)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma\\&=R^F_{\alpha k\beta l}+\left(C^{\mu}_{\nu\sigma;\rho}-C^{\mu}_{\nu\rho;\sigma}-C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon}+C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon}\right)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma\end{aligned} \tag{214}$$

We have already calculated $R^{\mu}_{\nu\sigma\rho}[e_\alpha]_\mu[e_k]^\nu[e_\beta]^\sigma[e_l]^\rho$ with linear order perturbation in section 8.3 so we can easily get

$$\begin{aligned} R^F_{0l0m} &= a_F^2(\eta_F)R^{\mu}_{\nu\sigma\rho}[e_0]_\mu[e_l]^\nu[e_0]^\sigma[e_m]^\rho \\ &= \frac{a_F^2(\eta_F)}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A)+\mathcal{H}A'\delta_{lm}+A_{,lm}-B'_{(l,m)}-\mathcal{H}B_{(l,m)}-C''_{lm}-\mathcal{H}C'_{lm}] \end{aligned} \tag{215}$$

$$\begin{aligned} R^F_{0lim} &= a_F^2(\eta_F)R^{\mu}_{\nu\sigma\rho}[e_0]_\mu[e_l]^\nu[e_i]^\sigma[e_m]^\rho \\ &= \frac{a_F^2(\eta_F)}{a^2}\Big[-(\mathcal{H}^2-\mathcal{H}')\delta_{li}(V_m-B_m)+(\mathcal{H}^2-\mathcal{H}')\delta_{lm}(V_i-B_i)-2\mathcal{H}\delta_{l[i}A_{,m]} \\ &\quad +B_{[i,m]l}+2C'_{l[i,m]}\Big] \end{aligned} \tag{216}$$

$$\begin{aligned} R^F_{iljm} &= a_F^2(\eta_F)R^{\mu}_{\nu\sigma\rho}[e_i]_\mu[e_l]^\nu[e_j]^\sigma[e_m]^\rho \\ &= \frac{a_F^2(\eta_F)}{a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A)+\mathcal{H}(\delta_{lm}B_{(i,j)}-\delta_{lj}B_{(i,m)}+\delta_{ij}B_{(l,m)}-\delta_{im}B_{(l,j)}) \\ &\quad +2\mathcal{H}(\delta_{l[m}C'_{j]i}+\delta_{i[j}C'_{m]l})+2C_{i[m,j]l}+2C_{l[j,m]i}] \end{aligned} \tag{217}$$

Now we are left to calculate the rest terms.

$$\begin{aligned} &\left(C^{\mu}_{\nu\sigma;\rho}-C^{\mu}_{\nu\rho;\sigma}\right)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= \Big(-\delta^\mu_\sigma\nabla_\rho\nabla_\nu\ln a_F+g_{\nu\sigma}g^{\mu\lambda}\nabla_\rho\nabla_\lambda\ln a_F+\delta^\mu_\rho\nabla_\sigma\nabla_\nu\ln a_F-g_{\nu\rho}g^{\mu\omega}\nabla_\sigma\nabla_\omega\ln a_F\Big)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= \Big(g_{\nu\sigma}g^{\mu\lambda}(\ln a_F)''[\tilde{e}_0]_\lambda[\tilde{e}_0]_\rho-g_{\nu\rho}g^{\mu\omega}(\ln a_F)''[\tilde{e}_0]_\omega[\tilde{e}_0]_\sigma\Big)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= \eta_{0\alpha}\eta_{0\beta}\delta_{kl}\mathcal{H}'_F \end{aligned} \tag{218}$$

$$\begin{aligned} &C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon}[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= (2\nabla_\rho\ln a_F\delta^\mu_\sigma\nabla_\nu\ln a_F+\delta^\mu_\nu\nabla_\rho\ln a_F\nabla_\sigma\ln a_F+\delta^\mu_\rho\nabla_\nu\ln a_F\nabla_\sigma\ln a_F \\ &\quad -g_{\nu\rho}g^{\epsilon\lambda}\nabla_\lambda\ln a_F\delta^\mu_\sigma\nabla_\epsilon\ln a_F-g_{\nu\rho}g^{\epsilon\lambda}\nabla_\lambda\ln a_F\delta^\mu_\epsilon\nabla_\sigma\ln a_F \\ &\quad -g_{\sigma\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F\delta^\epsilon_\nu\nabla_\rho\ln a_F-g_{\sigma\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F\delta^\epsilon_\rho\nabla_\nu\ln a_F \\ &\quad +g_{\nu\rho}g^{\epsilon\lambda}\nabla_\lambda\ln a_F g_{\sigma\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= \eta_{k\beta}\eta_{l\alpha}\mathcal{H}_F^2-\eta_{0\alpha}\eta_{0\beta}\delta_{kl}\mathcal{H}_F^2 \end{aligned} \tag{219}$$

$$\begin{aligned} &C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon}[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= (2\nabla_\sigma\ln a_F\delta^\mu_\rho\nabla_\nu\ln a_F+\delta^\mu_\nu\nabla_\sigma\ln a_F\nabla_\rho\ln a_F+\delta^\mu_\sigma\nabla_\nu\ln a_F\nabla_\rho\ln a_F \\ &\quad -g_{\nu\sigma}g^{\epsilon\lambda}\nabla_\lambda\ln a_F\delta^\mu_\rho\nabla_\epsilon\ln a_F-g_{\nu\sigma}g^{\epsilon\lambda}\nabla_\lambda\ln a_F\delta^\mu_\epsilon\nabla_\rho\ln a_F \\ &\quad -g_{\rho\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F\delta^\epsilon_\nu\nabla_\sigma\ln a_F-g_{\rho\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F\delta^\epsilon_\sigma\nabla_\nu\ln a_F \\ &\quad +g_{\nu\sigma}g^{\epsilon\lambda}\nabla_\lambda\ln a_F g_{\rho\epsilon}g^{\mu\omega}\nabla_\omega\ln a_F)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= \eta_{kl}\eta_{\beta\alpha}\mathcal{H}_F^2-\eta_{0\alpha}\eta_{0\beta}\delta_{kl}\mathcal{H}_F^2 \end{aligned} \tag{220}$$

Finally we find the conformal rescaling of the Riemann tensor gives

$$\begin{aligned} \tilde{R}^F_{\alpha k\beta l} &= R^F_{\alpha k\beta l}+\left(C^{\mu}_{\nu\sigma;\rho}-C^{\mu}_{\nu\rho;\sigma}-C^{\epsilon}_{\nu\rho}C^{\mu}_{\sigma\epsilon}+C^{\epsilon}_{\nu\sigma}C^{\mu}_{\rho\epsilon}\right)[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma \\ &= R^F_{\alpha k\beta l}+\eta_{0\alpha}\eta_{0\beta}\delta_{kl}\mathcal{H}'_F-\eta_{k\beta}\eta_{l\alpha}\mathcal{H}_F^2+\eta_{kl}\eta_{\beta\alpha}\mathcal{H}_F^2 \end{aligned} \tag{221}$$

By doing the conformal rescaling of the metric we obtain the Riemann tensor in the CFC.

$$\begin{aligned}\tilde{R}^F_{0l0m} &= R^F_{0l0m} + a_F^{-2}\delta_{lm}\mathcal{H}'_F \\ &= \frac{a_F^2(\eta_F)}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A) + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm}] \\ &\quad + \delta_{lm}\mathcal{H}'_F \\ &= -\frac{a_F^2(\eta_F)}{a^2}\mathcal{H}'\delta_{lm} + \delta_{lm}\mathcal{H}'_F + \mathcal{O}(\epsilon)\end{aligned} \tag{222}$$

$$\begin{aligned}\tilde{R}^F_{0lim} &= R^F_{0lim} - 0 \\ &= \frac{a_F^2(\eta_F)}{a^2}\Big[-(\mathcal{H}^2-\mathcal{H}')\delta_{li}(V_m - B_m) + (\mathcal{H}^2-\mathcal{H}')\delta_{lm}(V_i - B_i) - 2\mathcal{H}\delta_{l[i}A_{,m]} \\ &\qquad + B_{[i,m]l} + 2C'_{l[i,m]}\Big]\end{aligned} \tag{223}$$

$$\begin{aligned}\tilde{R}^F_{iljm} &= R^F_{iljm} - \delta_{im}\delta_{lj}\mathcal{H}_F^{-2} + \delta_{ij}\delta_{lm}\mathcal{H}_F^2 \\ &= \frac{a_F^2(\eta_F)}{a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A) + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) \\ &\qquad + 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}] - \delta_{im}\delta_{lj}\mathcal{H}_F^2 + \delta_{ij}\delta_{lm}\mathcal{H}_F^2 \\ &= 2\frac{a_F^2(\eta_F)}{a^2}\mathcal{H}^2\delta_{i[j}\delta_{m]l} - 2\delta_{i[j}\delta_{m]l}\mathcal{H}_F^2 + \mathcal{O}(\epsilon)\end{aligned} \tag{224}$$

Where ϵ is the parameter that quantifies the amplitude of the linear order perturbations and $\mathcal{O}(\epsilon)$ stands for a certain quantity of order ϵ.

We have found the conformal Riemann tensor under the conformal Fermi coordinates when choosing the Robertson-Walker metric with perturbations as the background. From the equation, we find that the CFC scale factor $a_F(\eta_F)$ is very important here. Our original purpose of setting up the CFC is to solve the problem of FNC that the Riemann tensor in NFC has the $\mathcal{H}^2$ and $\mathcal{H}'$ terms which limits the size of the patch can be covered by FNC. We are able to see if we simply set $a_F = a$, the conformal Riemann tensor in CFC is only 0 plus small perturbations. However, it is possible to define the CFC scale factor a_F differently.

11 Choice of the CFC scale factor

So far we have only said that $a_F(\eta_F)$ is a scale factor parameterized by the proper time. But we have not talked about anything related to the physical meaning of it. Now we will discuss the choice of this scale factor.

When applying to cosmology it is very natural for us to choose the scale factor a in the Robertson-Walker metric. Since in Robertson-Walker space-time, if we choose $a_F = a$, the conformal metric is simply $\tilde{g}_{\mu\nu} = a^{-2}g_{\mu\nu} = \eta_{\mu\nu}$. In this case, the $\mathcal{H}^2$ and $\mathcal{H}'$ in the conformal Riemann tensor in CFC are canceled. For the perturbed Robertson-Walker space-time, the conformal Riemann tensor in CFC is simply 0 plus some perturbations.

$$\tilde{R}^F_{0l0m} = 2\mathcal{H}'\delta_{lm}A + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm} \tag{225}$$

$$\begin{aligned}\tilde{R}^F_{0lim} = &-(\mathcal{H}^2-\mathcal{H}')\delta_{li}(V_m - B_m) + (\mathcal{H}^2-\mathcal{H}')\delta_{lm}(V_i - B_i) - 2\mathcal{H}\delta_{l[i}A_{,m]} \\ &+ B_{[i,m]l} + 2C'_{l[i,m]}\end{aligned} \tag{226}$$

$$\begin{aligned}\tilde{R}^F_{iljm} = &-4\mathcal{H}^2\delta_{i[j}\delta_{m]l}A + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) \\ &+ 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}\end{aligned} \tag{227}$$

Therefore, the CFC solves the patch limitation problem in the FNC. Since the conformal Riemann tensor in CFC is only perturbations the patch that can be studied with CFC reaches the supper-horizon size.

However, we can also define the CFC scale in a different way. For a local observer in the perturbed Robertson-Walker space-time. The scale factor a is a background expansion factor. The real expansion of the space observed by the local observer is not a. In fact, the real expansion factor will be the background scale factor a plus some perturbations.

First, we consider a couple of particles are free-falling in a certain region in the time-space which will give a bunch of time-like geodesics. We can study the expansion, shear, and rotation of the particles on these geodesics with respect to proper time τ. We define $B_{\mu\nu} = \nabla_\nu U^\mu$ which is the velocity shear tensor. This tensor $B_{\mu\nu}$ can be decomposed into the trace part, the trace-free symmetric part, and the antisymmetric part.

$$B_{\mu\nu} = \frac{1}{3}\vartheta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{228}$$

Where $P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$. The trace part ϑ contains the information on the expansion of the bunch of geodesics, σ contains the information on the shear, and ω contains the information on the rotation. Then we want to see the time evolution of the velocity shear tensor.

$$\begin{aligned}\frac{DB_{\mu\nu}}{D\tau} &= U^\sigma\nabla_\sigma B_{\mu\nu} = U^\sigma\nabla_\sigma\nabla_\nu U_\mu \\ &= U^\sigma\nabla_\nu\nabla_\sigma U_\mu + U^\sigma R^\lambda{}_{\mu\nu\sigma}U_\lambda \\ &= \nabla_\nu(U^\sigma\nabla_\sigma U_\mu) - \nabla_\nu U^\sigma\nabla_\sigma U_\mu + U^\lambda U^\sigma R_{\lambda\mu\nu\sigma} \\ &= -B^\sigma{}_\nu B_{\mu\sigma} + U^\lambda U^\sigma R_{\lambda\mu\nu\sigma}\end{aligned} \tag{229}$$

Since $U^\sigma\nabla_\sigma U_\mu = 0$ satisfies the geodesic equation.

We find $P^{\mu\nu}\frac{DB_{\mu\nu}}{D\tau} = \frac{D(P^{\mu\nu}B_{\mu\nu})}{D\tau}$. Where we have used the geodesic equation again. From this equation, we can get the well-known Raychaudhuri equation.

$$\frac{d}{d\tau}\vartheta = -\frac{1}{3}\vartheta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R^\mu_{\beta\mu\alpha}U^\beta U^\alpha \tag{230}$$

Working in the Conformal Fermi Coordinates we have

$$U^\mu_F = \frac{\partial x^\mu_F}{\partial\tau} = \frac{\partial x^\mu_F}{\partial\eta_F}\frac{\partial\eta_F}{\partial\tau} = \frac{1}{a_F(\eta_F)}[1,0,0,0] \tag{231}$$

Since in CFC the space-time is flat on the central geodesic. The Raychaudhuri equation can be written as

$$\begin{aligned}\frac{d}{d\tau}\vartheta &= -\frac{1}{3}\vartheta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - (R^F)^\mu_{\beta\mu\alpha}U^\beta_F U^\alpha_F \\ &= -\frac{1}{3}\vartheta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - \frac{1}{a_F^2}(R^F)^\mu_{0\mu0}\end{aligned} \tag{232}$$

Where $(R^F)^\mu_{0\mu0} = R^\nu_{\gamma\nu\theta}[\tilde{e}^\mu]_\nu[\tilde{e}_\mu]^\nu[\tilde{e}_0]^\gamma[\tilde{e}_0]^\theta$ is based on the global metric $g_{\mu\nu}$. Noted that $(\tilde{R}^F)^\mu_{0\mu0} = \tilde{R}^\nu_{\gamma\nu\theta}[\tilde{e}^\mu]_\nu[\tilde{e}_\mu]^\nu[\tilde{e}_0]^\gamma[\tilde{e}_0]^\theta$ is based on the conformal metric $\tilde{g}_{\mu\nu} = a_F^{-2}g_{\mu\nu}$. The relation between $(R^F)^\mu_{0\mu0}$ and $(\tilde{R}^F)^\mu_{0\mu0}$ is thus

$$a_F^{-2}(R^F)^\mu_{0\mu0} = a_F^{-2}(\tilde{R}^F)^\mu_{0\mu0} - 3(\dot{H}_F + H_F^2) \tag{233}$$

So that the Raychaudhuri equation in CFC is

$$\frac{d}{d\tau}\vartheta = -\frac{1}{3}\vartheta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - \frac{1}{a_F^2}(\tilde{R}^F)^\mu_{0\mu0} + 3(\dot{H}_F + H_F^2) \tag{234}$$

By defining $H_F = \frac{d \ln a_F}{d\tau} := \frac{1}{3}\vartheta$, the Raychaudhuri equation in CFC is reduced to

$$0 = -\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - \frac{1}{a_F^2}(\tilde{R}^F)^{\mu}_{0\mu 0} \tag{235}$$

Where $\vartheta = \nabla_\mu U^\mu$ is the velocity divergence. So, one way of interpreting our choice of a_F is that it brings the Raychaudhuri equation into the simple form of Eq.(235)

So far we have not considered any specific global metric. Considering the FLRW metric with perturbations, we can look at the velocity divergence ϑ in more detail. The global metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{236}$$

Where

$$g_{00} := -a^2(1+2A), \quad g_{0a} := -a^2 B_a, \quad g_{ab} := a^2\left(\delta_{ab} + 2C_{ab}\right) \tag{237}$$

The 4-velocity along the time-like central geodesic is

$$U^\mu = \frac{1}{a}\left(1 - A, V^a\right) \tag{238}$$

In the perturbed FLRW space-time ϑ is given by

$$\vartheta = \frac{1}{a}\left[3\mathcal{H} - 3\mathcal{H}A + C^{a\prime}_{\ a} + \partial_a V^a\right] \tag{239}$$

So that $H_F = \frac{1}{3}\vartheta$ is given by

$$H_F = \frac{d \ln a_F}{d\tau} = \frac{1}{a}\left[\mathcal{H} - \mathcal{H}A + \frac{1}{3}C^{a\prime}_{\ a} + \frac{1}{3}\partial_a V^a\right] = \frac{\mathcal{H}_F}{a_F} \tag{240}$$

After some calculations we find

$$\frac{d}{d\eta}\frac{a_F\left(\eta_F\right)}{a(\eta)} = -\frac{a_F\mathcal{H}}{a} + \frac{a_F\mathcal{H}_F}{a}\frac{d\eta_F}{d\eta} = -\frac{a_F\mathcal{H}}{a} + \frac{a_F\mathcal{H}_F}{a}\frac{d\eta_F}{d\tau}\frac{d\tau}{d\eta} = \frac{a_F}{a}\left[\frac{1}{3}C^{a\prime}_{\ a} + \frac{1}{3}\partial_a V_a\right] \tag{241}$$

Thus, we get the relation between the background scale factor and the CFC scale factor a_F under this definition. The CFC scale factor a_F now is the background scale factor a plus some perturbations.[43]

$$a_F = a + \mathcal{O}(\epsilon) \tag{242}$$

Thus, the conformal Riemann tensor in CFC is also order of ϵ. Therefore, using Eq.(212) we find the condition for the metric to be close to Minkowski is

$$\mathcal{O}(\tilde{R}^F_{\mu l \nu m} x^l_F x^m_F) \approx \mathcal{O}(\epsilon) x_F^2 < 1 \implies x_F < [\mathcal{O}(\epsilon)]^{-\frac{1}{2}} \tag{243}$$

Comparing to the case in FNC the scale of the x_F is much larger.

Part VI
Conclusions

In this thesis, we have constructed the Fermi normal coordinates (FNC) and the conformal Fermi coordinates (CFC). These are the coordinates of a free-falling observer that describes the neighborhood as a Minkowski space-time or a Robertson-Walker space-time up to corrections that grow with the square of the distance from the time-like geodesic of the free-falling observer.

Fermi normal coordinates are constructed on any background metric $g_{\mu\nu}$. We considered a free-falling observer who gives a time-like geodesic $x^\mu(\tau)$ which is parameterized by the proper time. Choose a point P on the geodesic as the origin of the coordinates. There exist four orthonormal tetrads that fix the coordinate axes there such that the metric on point P on the geodesic is in the normal form $\eta_{\mu\nu}$. By parallel transporting the four tetrads along the central geodesic, we make sure that the metric fixed by the tetrads on every point on the geodesic has the form $\eta_{\mu\nu}$. For a point Q with definite proper distance s_Q that is uniquely connected by another space like geodesic $x^\mu(s)$ from $x^\mu(s=0)=P$ the Fermi normal coordinates are defined by $x^i_F = a^i s_Q$. Where a^i is the components of the tangent vector on the space-like geodesic at point P when using the spacial tetrads as the base of the vector space. The global coordinates of the point Q can then be expanded around P with different orders of the Fermi normal coordinates.

In order to find the metric in FNC we further calculated the derivatives of the coordinate transformation $\frac{\partial x^\mu}{\partial x^\alpha_F}$. By using the coordinates transformation law of tensor $g^F_{\mu\nu}(Q) = \frac{\partial x^\alpha}{\partial x^\mu_F}\frac{\partial x^\beta}{\partial x^\nu_F} g_{\alpha\beta}(Q)$ we found the form of the metric in FNC for arbitrary background space-time.

$$\begin{aligned}
g^F_{00}(Q) &= \eta_{00} - R^F_{0l0m} x^l_F x^m_F \\
g^F_{0a}(Q) &= \eta_{0a} - \frac{2}{3} R^F_{0lam} x^l_F x^m_F \\
g^F_{ab}(Q) &= \eta_{ab} - \frac{1}{3} R^F_{albm} x^l_F x^m_F
\end{aligned} \tag{244}$$

We applied the FNC to the perturbed Robertson-Walker space-time. First, we found the four tetrads in this background and then calculated the Riemann tensor in FNC by $R^F_{\alpha\beta\gamma\delta} = [e_\alpha]^\mu_P [e_\beta]^\nu_P [e_\gamma]^\kappa_P [e_\delta]^\lambda_P R_{\mu\nu\kappa\lambda}$. Where The tetrads and the Riemann tensor are based on the Robertson-Walker metric with perturbations. To the linear order we found

$$\begin{aligned}
g^F_{00}(Q) &= \eta_{00} - \frac{1}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A) + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm}] x^l_F x^m_F \\
g^F_{0a}(Q) &= \eta_{0a} - \frac{2}{3a^2}[-(\mathcal{H}^2 - \mathcal{H}')\delta_{li}(V_m - B_m) + (\mathcal{H}^2 - \mathcal{H}')\delta_{lm}(V_i - B_i) - 2\mathcal{H}\delta_{l[i}A_{,m]} \\
&\quad + B_{[i,m]l} + 2C'_{l[i,m]}] x^l_F x^m_F \\
g^F_{ab}(Q) &= \eta_{ab} - \frac{1}{3a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A) + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) \\
&\quad + 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}] x^l_F x^m_F
\end{aligned} \tag{245}$$

However, Robertson Walker metric brings H^2 to the Riemann tensor which brings a problem to the size of the patch that can be covered by FNC because for $x_F > H^{-1}$ the deviation of FNC metric from Minkowski space is too large. Thus the scale that can be studied is limited. This is the motivation for us to construct the conformal Fermi coordinates.

The construction of the conformal Fermi coordinates is similar to the construction of the Fermi normal coordinates. We defined a 'conformal metric' $\tilde{g}_{\mu\nu}(x) \equiv a_F^{-2}(\tau) g_{\mu\nu}(x)$ to take the scale factor out of the global metric around the time-like central geodesic. Again we found the four tetrads but this time based on the conformal metric $[\tilde{e}_a]^\mu = a_F(\tau)[e_a]^\mu$. The CFC coordinates x^i_F are defined based on conformal metric $x^i_F = \beta^i \tilde{s}_Q$. Where $\tilde{s}_Q$ is the conformal proper distance. Following the same procedures we have done for NFC, we were able to find the metric in CFC

$$
\begin{aligned}
g^F_{00}(Q) &= a^2_F(\eta_F)\left[\eta_{00} - \left.\tilde{R}^F_{0k0l}\right|_P x^k_F x^l_F\right], \\
g^F_{0a}(Q) &= a^2_F(\eta_F)\left[\eta_{0a} - \frac{2}{3}\left.\tilde{R}^F_{0kal}\right|_P x^k_F x^l_F\right], \\
g^F_{ab}(Q) &= a^2_F(\eta_F)\left[\eta_{ab} - \frac{1}{3}\left.\tilde{R}^F_{akbl}\right|_P x^k_F x^l_F\right].
\end{aligned}
\tag{246}
$$

We again applied the perturbed Robertson-Walker metric to the background. Calculated the conformal Riemann tensor in CFC with $\tilde{R}^F_{\alpha k\beta l} = \tilde{R}^\mu_{\nu\rho\sigma}[\tilde{e}_\alpha]_\mu[\tilde{e}_k]^\nu[\tilde{e}_\beta]^\rho[\tilde{e}_l]^\sigma$ where $\tilde{R}^\mu_{\nu\rho\sigma}$ is the conformal Riemann tensor based on the perturbed Robertson-Walker metric. To the linear order we found

$$
\begin{aligned}
\tilde{R}^F_{0l0m} &= R^F_{0l0m} + a_F^{-2}\delta_{lm}\mathcal{H}'_F \\
&= \frac{a^2_F(\eta_F)}{a^2}[-\mathcal{H}'\delta_{lm}(1-2A) + \mathcal{H}A'\delta_{lm} + A_{,lm} - B'_{(l,m)} - \mathcal{H}B_{(l,m)} - C''_{lm} - \mathcal{H}C'_{lm}] \\
&\quad + \delta_{lm}\mathcal{H}'_F \\
\tilde{R}^F_{0lim} &= R^F_{0lim} - 0 \\
&= \frac{a^2_F(\eta_F)}{a^2}\left[-(\mathcal{H}^2 - \mathcal{H}')\delta_{li}(V_m - B_m) + (\mathcal{H}^2 - \mathcal{H}')\delta_{lm}(V_i - B_i) - 2\mathcal{H}\delta_{l[i}A_{,m]}\right. \\
&\qquad \left. + B_{[i,m]l} + 2C'_{l[i,m]}\right] \\
\tilde{R}^F_{iljm} &= R^F_{iljm} - \delta_{im}\delta_{lj}\mathcal{H}_F^{-2} + \delta_{ij}\delta_{lm}\mathcal{H}^2_F \\
&= \frac{a^2_F(\eta_F)}{a^2}[2\mathcal{H}^2\delta_{i[j}\delta_{m]l}(1-2A) + \mathcal{H}(\delta_{lm}B_{(i,j)} - \delta_{lj}B_{(i,m)} + \delta_{ij}B_{(l,m)} - \delta_{im}B_{(l,j)}) \\
&\qquad + 2\mathcal{H}(\delta_{l[m}C'_{j]i} + \delta_{i[j}C'_{m]l}) + 2C_{i[m,j]l} + 2C_{l[j,m]i}] - \delta_{im}\delta_{lj}\mathcal{H}^2_F + \delta_{ij}\delta_{lm}\mathcal{H}^2_F
\end{aligned}
\tag{247}
$$

By choosing the CFC scale factor a_F to be the background scale factor a or a plus some perturbations we are able to leave the conformal Riemann tensor in CFC only with perturbations terms. Thus, the CFC solves the patch limitation problem in the FNC. Since the conformal Riemann tensor in CFC is only perturbations the patch that can be studied with CFC reaches the supper-horizon size.

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