SOFT GLUON FACTORIZATION AT TWO LOOPS IN FULL COLOR

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Zurich Particle Physics Seminar

arXiv: 1912.09370 with Dixon, Herrmann, Zhu.

Factorization of scattering amplitudes

When external particles are unresolved, gauge theory amplitudes factorize into lower-point amplitudes multiplied by a universal emission factor, e.g. splitting amplitudes, soft-gluon emission factors.

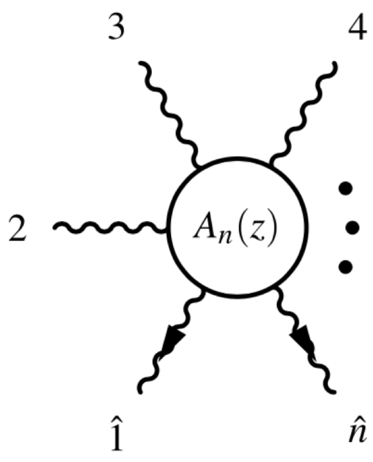
- The emission factors are typically simple and nice, a good way to probe analytic properties of the multi-point amplitudes.
- Capture phase-space infrared singularities, ingredients to IR subtraction scheme.

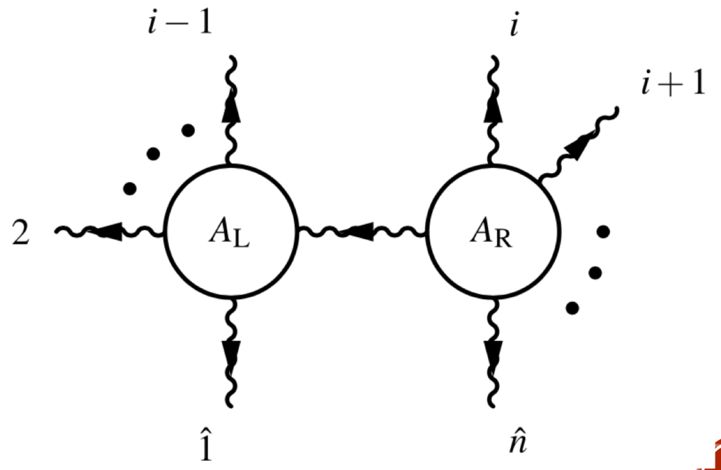
Recent progress at N^3LO: e.g. [Catani, Colferai, Torrini (2019), Del Duca, Duhr, Haindl, Lazopoulos, Michel (2019-20), Catani, de Florian, Rodrigo (2019);Zhu (2020)]

Tree level factorization:

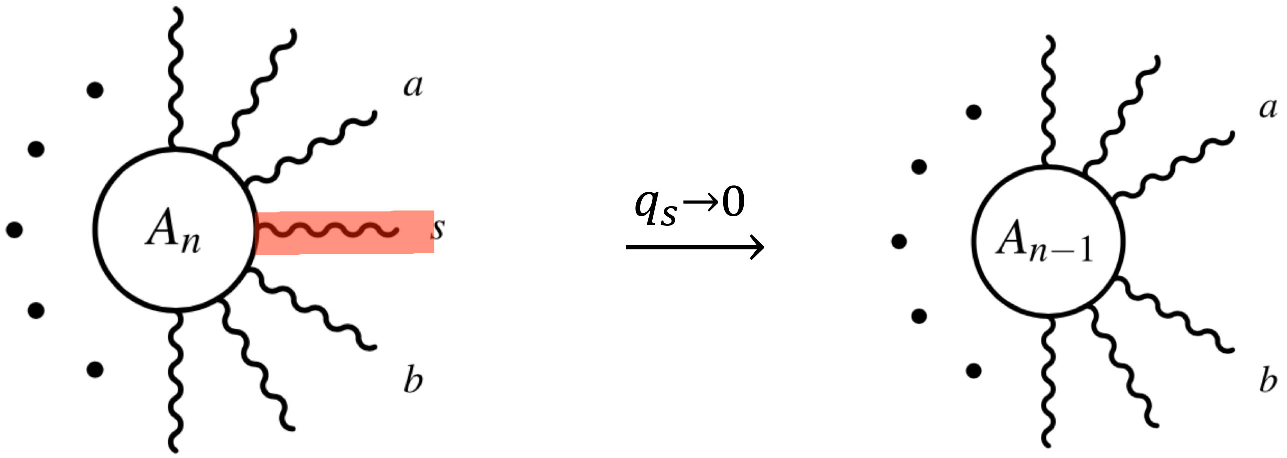
$$A_{tree}(1,\dots,n) \sim \sum_{\lambda} A_{tree}\left(i,\dots,j,P^{\lambda}\right)\frac{1}{P_{i,j}^{2}}\, A_{tree}\left(P^{\lambda},j+1,\dots,i-1\right)$$

color-ordered amplitudes have poles when region momenta $P_{i,j} := p_i + p_{i+1} + \cdots + p_j$ go on shell. At leading power as $P_{i,j}^2 \to 0$, they factorize into product of lower-point amplitudes.





Soft gluon factorization



$$\times \; S\,(s^{\pm}; \{1, \ldots, n-1\}\,)$$

S depend on the momentum and helicities of the soft gluon, independent of the helicities and particle types of the others

$$S^{tree}(s^+; a, b\,) = \frac{\langle a\, b\rangle}{\langle a\, q\rangle\langle q\, b\rangle} \qquad S^{tree}(s^-; a, b\,) = -\frac{[a\, b]}{[a\, q][q\, b]}$$

(Tree-level) soft emission factor is a sum of gauge invariant dipoles

- Dipole formula describes the planar limit of higher-loop amplitudes in soft limit

known up to 2-loop order

Anastasiou, Bern, Dixon, Kosower [0309040].

Duhr, Gehrmann [1309.4393] Li, Zhu [1309.4941]

- Dipole formula needs to be modified for multi-parton scattering processes

Quadruple correlation in three loop soft anomalous dimension

Almelid, Duhr, Gardi [1507.00047],
Almelid, Duhr, Gardi, McLeod, White, [1706.10162].

Soft gluon emission from Wilson lines

$S^{(2)}(q,\{p_i\})$ can be extracted from 5-pt amplitude 1+2 → 3+4+q :

$$\left|A_5^{(2)}\right\rangle \to S_\pm^{a,(2)}(q;\{p_i\})\left|A_4^{(0)}\right\rangle + S_\pm^{a,(1)}(q;\{p_i\})\left|A_4^{(1)}\right\rangle + S_\pm^{a,(0)}(q;\{p_i\})\left|A_4^{(2)}\right\rangle$$

Or directly obtained from Wilson-line matrix element

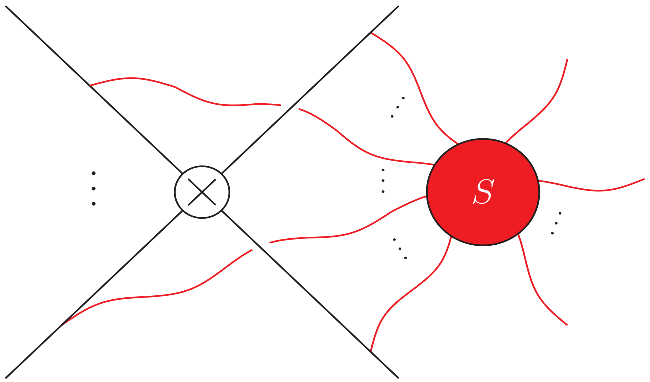
$$\langle q;a;\pm|\,Y_1\cdots Y_n|\,0\rangle = S_a^\pm(q,\{n_i\})\,\underline{\langle 0|\,Y_1\cdots Y_n|\,0\rangle}$$

= 1 in pure dim-Reg

$$Y_j(x) := \mathrm{P}\exp ig\int_0^\infty n_j\cdot A^a T^a\,(x+s\,n_j)\,ds$$

Represent classical sources traveling in a particular direction $\vec{n}_j := \frac{\vec{p}_j}{p^0}$

$S\,(X_s,\{n_i\}) =$



Feige, Schwartz [1403.6472].

invariance under recalling of momenta of classical sources : S (q) depends on one energy scale (the soft gluon energy) , and the angles between the directions of external momenta $\vec{n}_q, \{\vec{n}_i, \vec{n}_j, \overrightarrow{n_k} \ldots .\}$

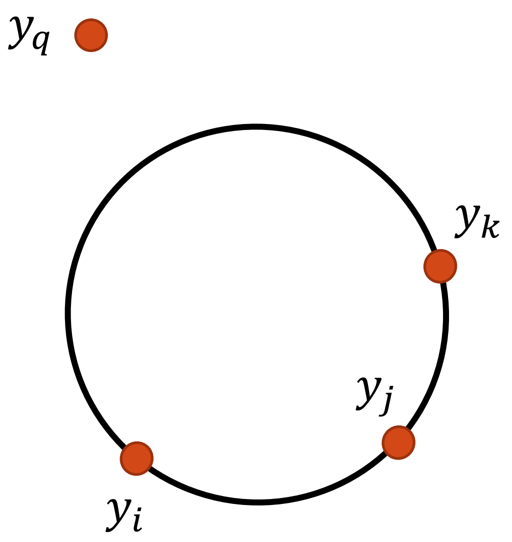
Symmetries and kinematics

Stereographic projection:

$$n^{\mu} = \left(1,\ \frac{y+\bar{y}}{1+y\,\bar{y}},\frac{-i\,(y-\bar{y})}{1+y\bar{y}}\ ,\frac{1-y\bar{y}}{1+y\bar{y}}\right)$$

Unit 2-sphere mapped onto y-plane,
Lorentz symmetry → global SL(2, C)

Through conformal boost $z \coloneqq \frac{(y-y_i)\,(y_j-y_k)}{(y-y_j)\,(y_i-y_k)}$



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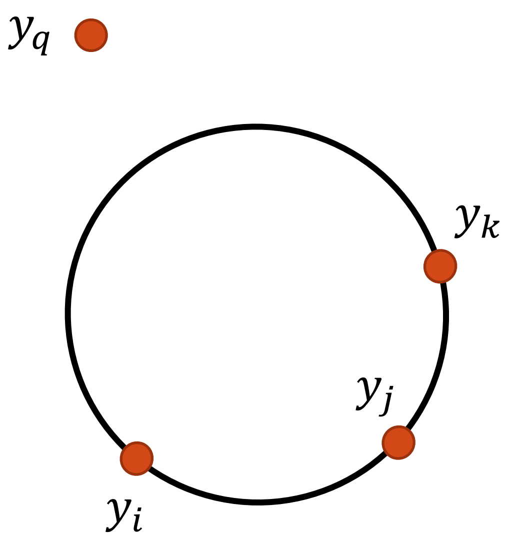
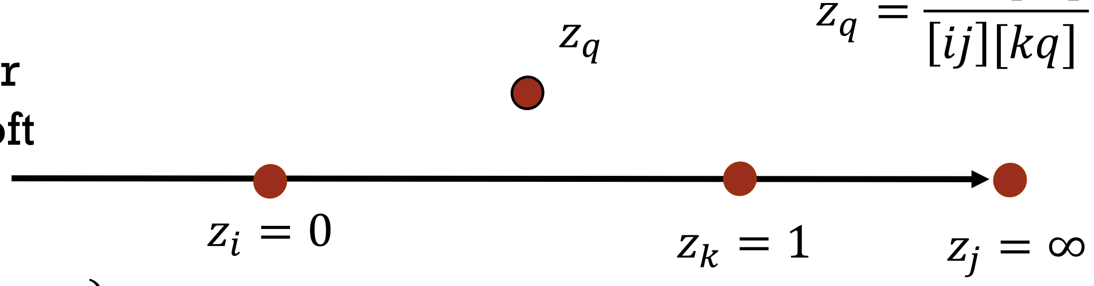


Through conformal boost $z \coloneqq \dfrac{(y-y_i)\,(y_j-y_k)}{(y-y_j)\,(y_i-y_k)}$

$$(y_i, y_q, y_k, y_j) \mapsto (0, z_q, 1, \infty)$$

$$z_q = \frac{\langle iq\rangle\langle kj\rangle}{\langle ij\rangle\langle kq\rangle}\ ,\qquad \bar{z}_q = \frac{[iq][kj]}{[ij][kq]}$$

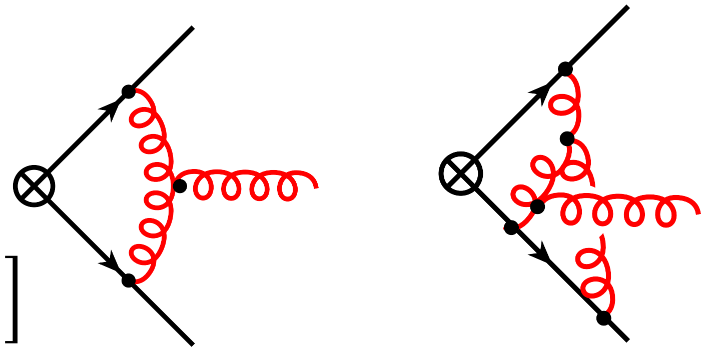


Number of independent kinematic variables for process with n external particles including 1 soft gluon: $2(n-3)+1$

overall energy scale : $x_{ij} \coloneqq \dfrac{(-s_{ij})}{(-s_{iq})(-s_{qj})}$

General structure of soft factorization at higher-loop orders

$$S = S_{dipole} + S_{tripole} + S_{quadruple} + \ldots$$



$$S_{dipole}(q,\{i,j\}) = S_{tree}\left[1 + a\,V_{ij}^{q}\,C_1(\epsilon) + [a\,V_{ij}^{q}]^2\,C_2(\epsilon) + \ldots\right]$$

$$V_{ij}^{q} := \left[\frac{\mu^2(-s_{ij})}{(-s_{iq})(-s_{qj})}\right]^{\epsilon}, \quad s_{ab} = \langle ab\rangle[ba] = -|p_a\cdot p_b|e^{-i\pi\lambda_{ab}}$$

λ_{ab}=1 both incoming/outgoing
λ_{ab}=0, otherwise

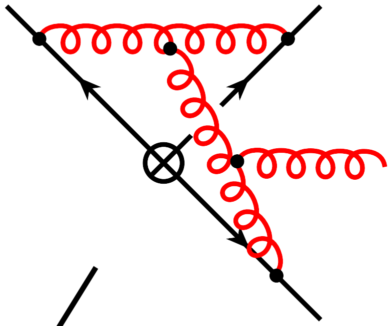
$$C_1(\epsilon) = -\frac{1}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon^2} - \frac{\zeta_2}{2} + \epsilon\frac{7}{3}\zeta_3 + \ldots$$

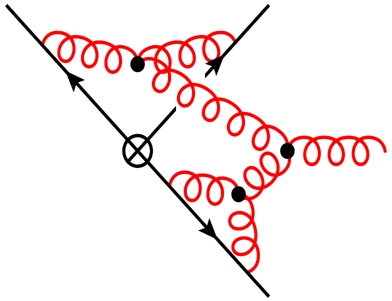
Uniform transcendental weight

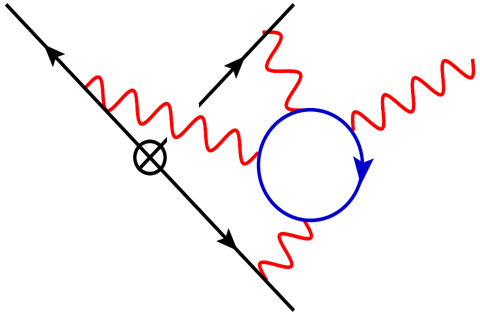
$$C_2(\epsilon) = C_A\,B_1 + T_R N_f\,B_2 + C_A\,N_s\,B_3$$

$T_R \to \frac{C_A}{2}$, $N_f \to 4$, $N_s \to 6$ agrees with planar N=4 SYM

2-loop and beyond





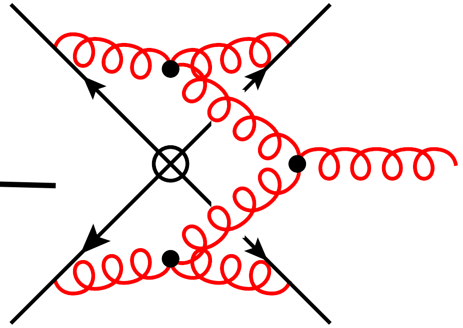


$$S_{tripole}\,(q,\{i,j,k\}) = S_{tree}\left[[a\,V_{ij}^{q}]^2\,F_2\left(\epsilon, z_q, \bar{z}_q\right) + [a\,V_{ij}^{q}]^3\,F_3\left(\epsilon, z_q, \bar{z}_q\right) + \ \ldots\right]$$

3-loop correction induced by fermion loop (quartic Casimir, quantum correction in abelian theory)

At the lowest perturbative order, no dependence on the matter content, have uniform weight property as in N=4 SYM

3-loop and beyond



$$S_{quad}\,(q,\{i,j,k,l\}) = S_{tree}\left[[a\,V_{ij}^{q}]^3\,G_3\left(\epsilon, z_q, \bar{z}_q, z_{ijkl}, \bar{z}_{ijkl}\right) + \ \ldots\right]$$

New structure at two-loop order

$$z_k^{ij} = \frac{\langle iq\rangle\langle kj\rangle}{\langle ij\rangle\langle kq\rangle}$$

We obtained the first correction to dipole formula at two-loop order in full color: a tripole emission factor

$$S^{+(2)}_{a,ijk} = V^2_{q,ij}\, f_{aa_kb}\, f_{ba_ia_j}\; T_i^{a_i}\, T_j^{a_j}\, T_k^{a_k} \left[\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F\left(z_k^{ij},\epsilon\right) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F\left(z_k^{ji},\epsilon\right) \right]$$

What can we learn from the result?

- Universal analytic properties (symbol alphabat, location of branch cut)

--- constraints for higher-loop amplitude (bootstrap)

- Integrands for phase-space integrals

--- N^3LO IR subtraction programs

--- Resummation of physical observables

- How does the soft gluon talk to the incoming vs. outgoing hard particles ?

---- conceptual issue with factorization of hadronic cross section at the LHC

Non-trivial absorbitive part of loop integrals starts playing a role at N3LO. Could spoil universality of collinear singularity

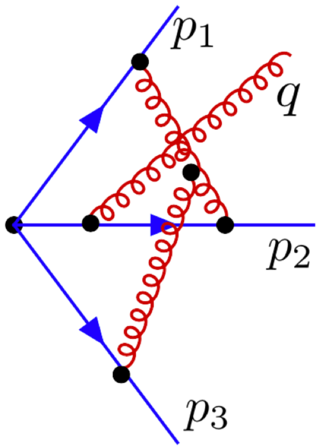
Catani, de Florian, Rodrigo [1112.4405]

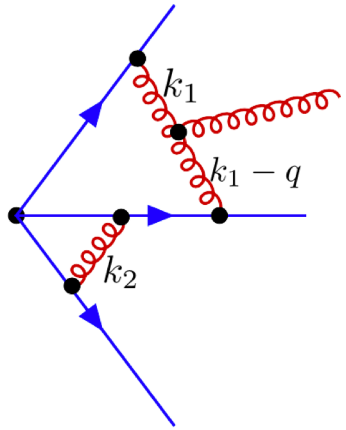
Forshaw, Seymour, Siodmok [1206.6363].

TWO-LOOP TRIPOLE EMISSION FACTOR

Regularization scheme:
Light-like Wilson line
$d = 4 - 2\epsilon$

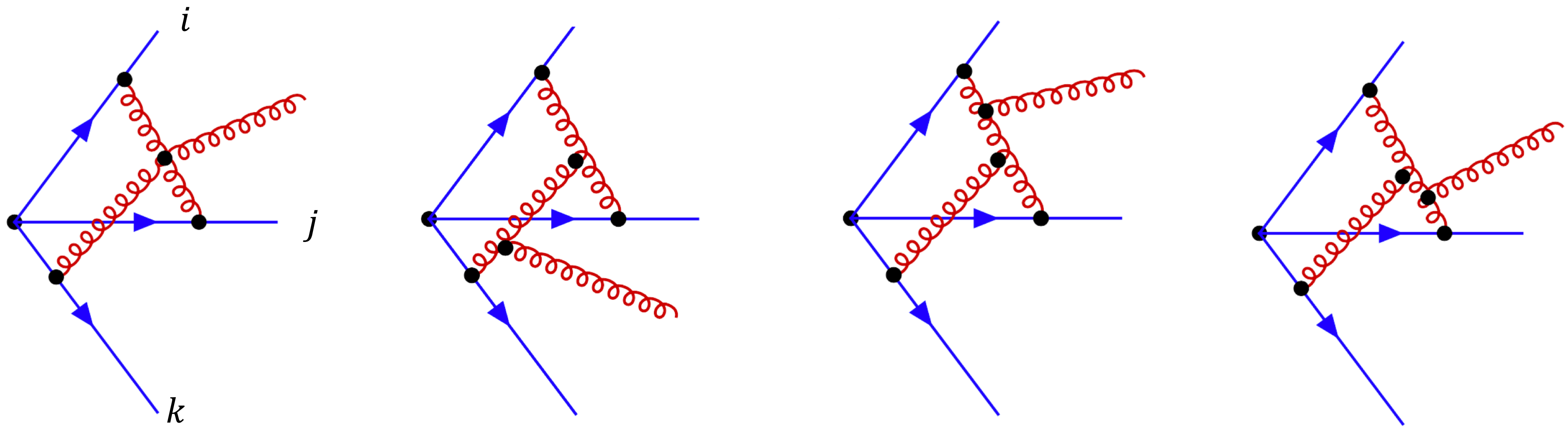
diagrams vanish if
1) depend on q only through q.p2
2) contains a scale-less sub-loop





Maximally non-abelian feynman diagrams

A,B belong to an integral family symmetric w.r.t $i \leftrightarrow$ j define a tripole (i,j,k)



A B C D

Two-loop dipole family

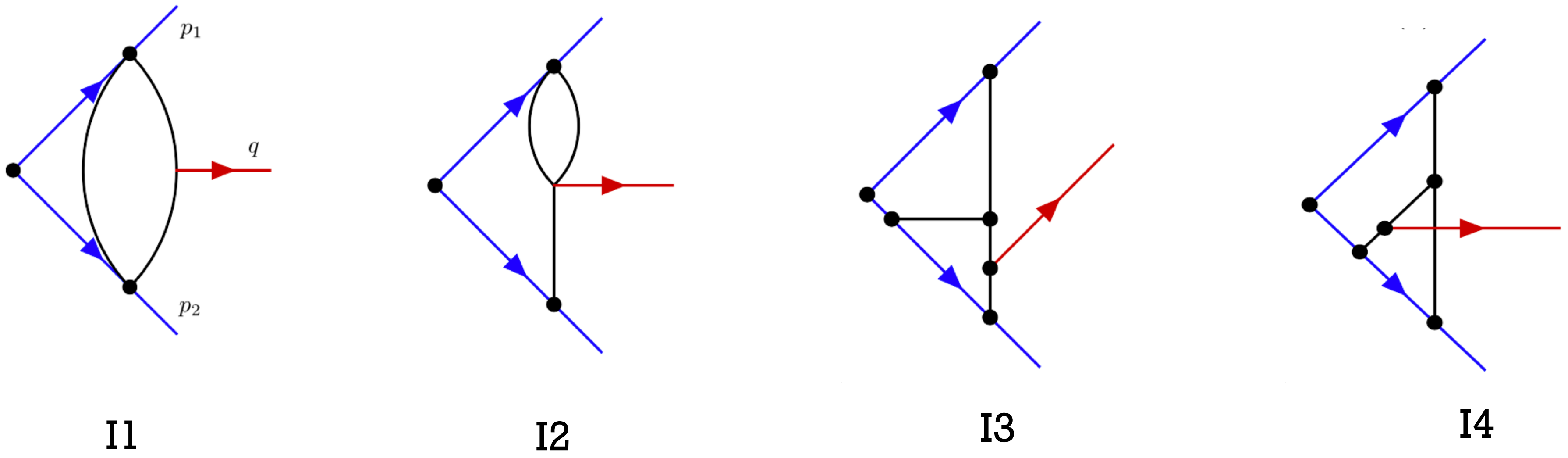
Two hard external partons, e.g. e+e- → qqbar:

$$S^{(2)}_{a,+}(q) = \left(V^{q}_{ij}\right)^2 f_{abc}T^b_i T^c_j \, C_2(\epsilon) \frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$

$$C_2(\epsilon) = C_A^2 \, B_1 \;\; + C_A N_s \, B_2 \;\; + C_A \, N_f \, B_3$$

Only planar contributions

Master integrals



1309.4941

In the multi-parton scattering process, non-planar contribution from I4 should cancel with the tripole diagrams

Differential equations for the two-loop tripole family

External kinematics $\frac{(-s_{ij})}{(-s_{iq})(-s_{qj})} := 1, \quad \frac{(-s_{ik})}{(-s_{iq})(-s_{qk})} := u\,, \quad \frac{(-s_{jk})}{(-s_{jq})(-s_{qk})} := v\,.$

8 Master integrals $d\,\vec{f} = d\,A\,(\epsilon, u, v)\,\vec{f}$

Differential equation contains logarithmic singularities at
$u = 0, v = 0, \;\Delta := 1 - 2\,u \; - 2\,v + (u - v)^2 = 0$

DE can be brought into canonical form

$$d\,\vec{g} = \epsilon \sum_i d \ln \alpha_i\,(z, \bar{z})\;\; B_i\,\vec{g},$$

$$\alpha := \{z_k^{ij}, 1 - z_k^{ij}, \bar{z}_k^{ij}, 1 - \bar{z}_k^{ij}, z_k^{ij} - \bar{z}_k^{ij}\}$$

$$z_k^{ij} := \frac{\langle iq\rangle\langle kj\rangle}{\langle ij\rangle\langle kq\rangle}\,, \qquad \bar{z}_k^{ij} := \frac{[iq][kj]}{[ij][kq]}$$

$$u = (1 - z_k^{ij})(\,1 - \,\bar{z}_k^{ij}), \qquad v = z_k^{ij}\,\bar{z}_k^{ij}\,.$$

$$\sqrt{\Delta} = z - \bar{z} = 4i\,\frac{\epsilon\big(p_i, p_j, p_k, q\big)}{s_{ij}\,s_{kq}}$$

Real-analyticity on the Euclidean sheet

In Euclidean region , i.e. all $x_{ij} := \frac{(-s_{ij})}{(-s_{iq})(-s_{qj})}$>0, the master integrals are real-analytic.

- $\overline{F(z,\bar{z})} = F(\bar{z},z), \quad \bar{z} = z^*$
- branch cut on the complex z-plane cancel

  logarithms in $z\,\bar{z},\ (1-z)(1-\bar{z})$ correspond to physical singularities in collinear limit:

  $$q \parallel p_i, p_j, \mathrm{p_k}, \qquad \mathrm{z} \to 0, 1, \infty$$

- up to $O(\epsilon^0)$, 4 letters $\quad \alpha := \{z\,, 1-z, \bar{z}, 1-\bar{z}\}$

Function space of the final answer is covered by Simple-valued Harmonic Polylogarithms :

$$\partial_z L_{w_0,\vec{w}} := (-1)^{w_0} \frac{1}{z-w_0} L_{\vec{w}},$$

$$L_{0^n} := \frac{1}{n!}\log^n(z\,\bar{z}), L_1 := -\log\big((1-z)(1-\bar{z})\big), \quad L_{\vec{w}} = 0,\ \forall\, \vec{w} \neq \vec{0},\ at\ z = 0.$$

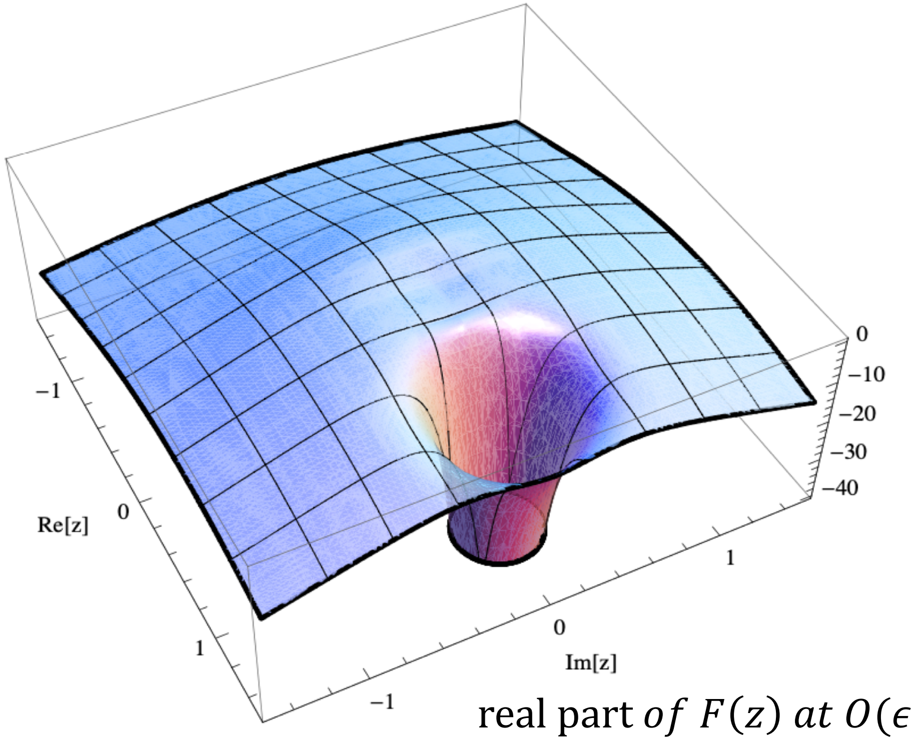
Final result for the (i,j,k) tripole:

$$S^{+(2)}_{a,ijk} = V^2_{q,ij}\, f_{aa_kb}\, f_{ba_ia_j}\; T_i^{a_i}\, T_j^{a_j}\, T_k^{a_k} \left[\, \frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} F\left(z_k^{ij},\epsilon\right) - \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} F\left(z_k^{ji},\epsilon\right) \,\right]$$

Symmetric under exchange of $i \leftrightarrow j\,,\quad z \leftrightarrow (1-z)$.

$(z_k^{ji} := 1 - z_k^{ij})$

$$F(z,\bar{z},\varepsilon) = \frac{1}{\epsilon^2} L_0L_1 + \frac{1}{3\,\epsilon}\left(L_1^2L_0 - 2\,L_0L_1^2\right) - L_1\left(\frac{2}{9} L_0L_1 + \frac{1}{3} L_0^2L_1 + \frac{13}{18} L_0L_1^2 + \frac{7}{12} L_1^3\right) + +\zeta_2(2L_{0,1} - L_0L_1) + \frac{40}{3}\zeta_3 L_1 + O(\varepsilon)$$



real part of $F(z)$ at $O(\epsilon^0)$

In the collinear limit

$q \parallel p_i,\qquad F\left(z_k^{ij},\bar{z}_k^{ij}\right) \xrightarrow{z,\bar{z}\to 0} 0.$

$q \parallel p_j \; or \; p_k,\qquad F\left(z_k^{ij},\bar{z}_k^{ij}\right) \xrightarrow{z,\bar{z}\to 1 \; or\; \infty} \infty\,.$

sum over 6 permutations among the Wilson lines

$$(i,j,k) \leftrightarrow (j,k,i) \leftrightarrow (k,i,j)$$

$$z \leftrightarrow \frac{z}{z-1} \leftrightarrow \frac{1}{1-z}$$

$$-\frac{1}{4}\sum_{i\neq k\neq j} \boldsymbol{S}^{+,(2)}_{a,ikj} = -\frac{1}{4}\sum_{\substack{\text{tripoles}\\ \{i,j,k\}}} \boldsymbol{S}^{+,(2)}_{a,\{i,j,k\}}$$

alternative definition of the tripole in terms of unordered tuple {i,j,k}

4 independent color and kinematic structures

$$\boldsymbol{S}^{+,(2)}_{a,\{i,j,k\}} = 2\Big(\boldsymbol{S}^{+,(2)}_{a,ikj} + \boldsymbol{S}^{+,(2)}_{a,kji} + \boldsymbol{S}^{+,(2)}_{a,jik}\Big)$$

$$= 2\,\boldsymbol{T}_i^{a_i}\boldsymbol{T}_j^{a_j}\boldsymbol{T}_k^{a_k}\left\{\frac{\langle ik\rangle}{\langle iq\rangle\langle qk\rangle}(V^q_{ik})^2\Big[f^{aa_jb}f^{ba_ia_k}D_1(z,\bar z) + f^{aa_ib}f^{ba_ka_j}D_2(z,\bar z)\Big]\right.$$

$$\left. + \{i\leftrightarrow j\}\right\}$$

Suppressed in all three collinear limits on Euclidean sheet

$$D_1(z,\bar z) = u^{-2\epsilon}F(z,\bar z) + F\left(\frac{-z}{1-z},\frac{-\bar z}{1-\bar z}\right)$$

$$D_2(z,\bar z) = u^{-2\epsilon}F(z,\bar z) - \left(\frac{u}{v}\right)^{-2\epsilon}\left[F\left(\frac{1}{z},\frac{1}{\bar z}\right) - F\left(\frac{1-z}{-z},\frac{1-\bar z}{-\bar z}\right)\right]$$

$$q \parallel p_i \text{ or } p_k,\quad D_i(z,\bar z) \xrightarrow{z,\bar z\to 0 \text{ or } \infty} 0.$$

$$q \parallel p_j,\quad D_i(z,\bar z) \xrightarrow{z,\bar z\to 1} \infty.$$

The epsilon poles come from the exponetiation of soft divergence $\frac{1}{\epsilon}\gamma_K \sum \log(\frac{-|s_{ij}|e^{-i\pi\lambda_{ij}}}{\mu^2})$

Final answer in terms of SVHPLs:

$$D_1(z) = -\frac{1}{\epsilon^2}(\mathcal{L}_1)^2 - \frac{1}{\epsilon}(\mathcal{L}_1)^3 - \frac{7}{12}(\mathcal{L}_1)^4 + 4\mathcal{L}_{1,0,1,0} + 2\mathcal{L}_{1,0,1,1} + 2\mathcal{L}_{1,1,1,0}$$

$$D_2(z) = \frac{1}{\epsilon^2}\mathcal{L}_0\mathcal{L}_1 + \frac{1}{\epsilon}\mathcal{L}_0(\mathcal{L}_1)^2 + \frac{2}{3}\mathcal{L}_0(\mathcal{L}_1)^3 + 6\,\zeta_2\left(\mathcal{L}_{0,1} - \mathcal{L}_{1,0}\right) + 2\left(\mathcal{L}_{0,0,0,1} - \mathcal{L}_{0,0,1,0} + \mathcal{L}_{0,1,0,0} + \mathcal{L}_{0,1,0,1} - \mathcal{L}_{1,0,0,0}\right).$$

The symbol level cross check :
matches with two-loop five-point amplitudes in N=4 SYM in the limit $p_5 \to 0$

Abreu, Dixon, Herrmann, Page, Zeng [1812.08941]
Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia [1812.11057]

$$s_{12} = x[1];\; s_{23} = x[2]\,x[4];$$

$$s_{34} = x[1]\left(x[4] - \frac{x[3](1 - x[4])}{x[2]}\right) + x[3]\,(x[4] - x[5]);$$

$$s_{45} = x[2](x[4] - x[5]);\; s_{15} = x[3]\,(1 - x[5]);$$

In the soft limit d-> 0,

$$x[1] \to s, \quad x[2] \to s\,x, \quad x[3] \to -s\,x/(1 - z),$$

$$x[4] \to 1 + d\left(\frac{x + \bar{z}}{1 - \bar{z}}\right), \quad x[5] \to 1 + d\,(1 + \frac{x + \bar{z}}{1 - \bar{z}})$$

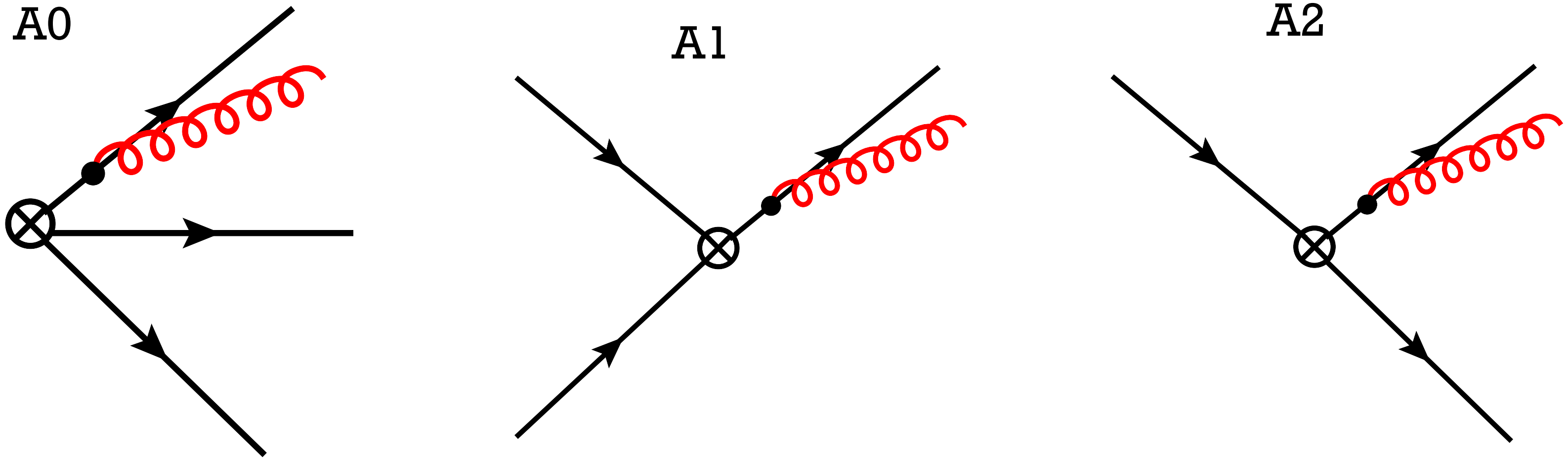
Analytic continuation into physical regions

$\bar{z} := z^*, \quad s_{ab} = -|p_a \cdot p_b| e^{-i\pi\lambda_{ab}}$

λ_{ab}=1 both incoming/outgoing
λ_{ab}=0, otherwise

$$\frac{s_{ik}s_{qj}}{s_{ij}s_{qk}} := u_k^{ij}\,, \quad \frac{s_{jk}s_{iq}}{s_{ij}s_{qk}} := v_k^{ij}\,.$$

Region	Kinematics	analytic continuation rule	
A_0	all outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert$
A_1	j,k incoming, q,i outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert e^{-2i\pi}$
A_2	i incoming, q,j,k outgoing	$u_k^{ij} \to \lvert u_k^{ij}\rvert$	$v_k^{ij} \to \lvert v_k^{ij}\rvert$



Analytic continuation in A_1 region requires taking the monodromy of SVHPLs at z=0.

$$D_i(z,\bar{z})|_{A_1} = D_i(z,\bar{z})|_{A_0} + \mathrm{disc}_{A_1} D_i(z,\bar{z}) \qquad \mathrm{disc}_{A_1} D_i(z,\bar{z}) = \underset{z \to z\, e^{-2\pi \mathrm{i}}}{\mathrm{disc}} [D_i(z,\bar{z})]$$

Starting from weight 1, build the analytic continuation for higher weight SVHPLs by requiring consistency with the differential equations.

$$d\; disc_{A_1}\, L_w(z) = disc_{A_1}\; d\, L_w\,(z); \qquad disc_{A_1} L_0 = -2\pi\, i, \quad disc_{A_1}\, L_1 = 0.$$

$disc_{A_1} D_1(z),\ disc_{A_1} D_1(1-z),\ disc_{A_1} D_2(z),\ disc_{A_1} D_2(1-z)$ are given by weight-3 classical polylogarithms

$$disc_{A_1} D_2(1-z\,) - \frac{1}{2} disc_{A_1} D_1(z) = -4\pi^2 \left\{ \frac{\log|1-z|^2}{\epsilon} - 2\log|z|^2 \log|1-z|^2 \right\}$$
$$+ 2\mathrm{i}\pi \left\{ \frac{\log|1-z|^2}{\epsilon^2} - \frac{2}{\epsilon} \log|z|^2 \log|1-z|^2 - \frac{1}{12} \log^3|1-z|^2 + 2\log^2|z|^2 \log|1-z|^2 - 16\zeta_2 \log|1-z|^2 + \frac{1}{4} \log\left(\frac{1-z}{1-\bar{z}}\right) \left[\log^2\left(\frac{1-z}{1-\bar{z}}\right) + 4\pi^2\right] \right\}$$

Single-valuedness in A1 region

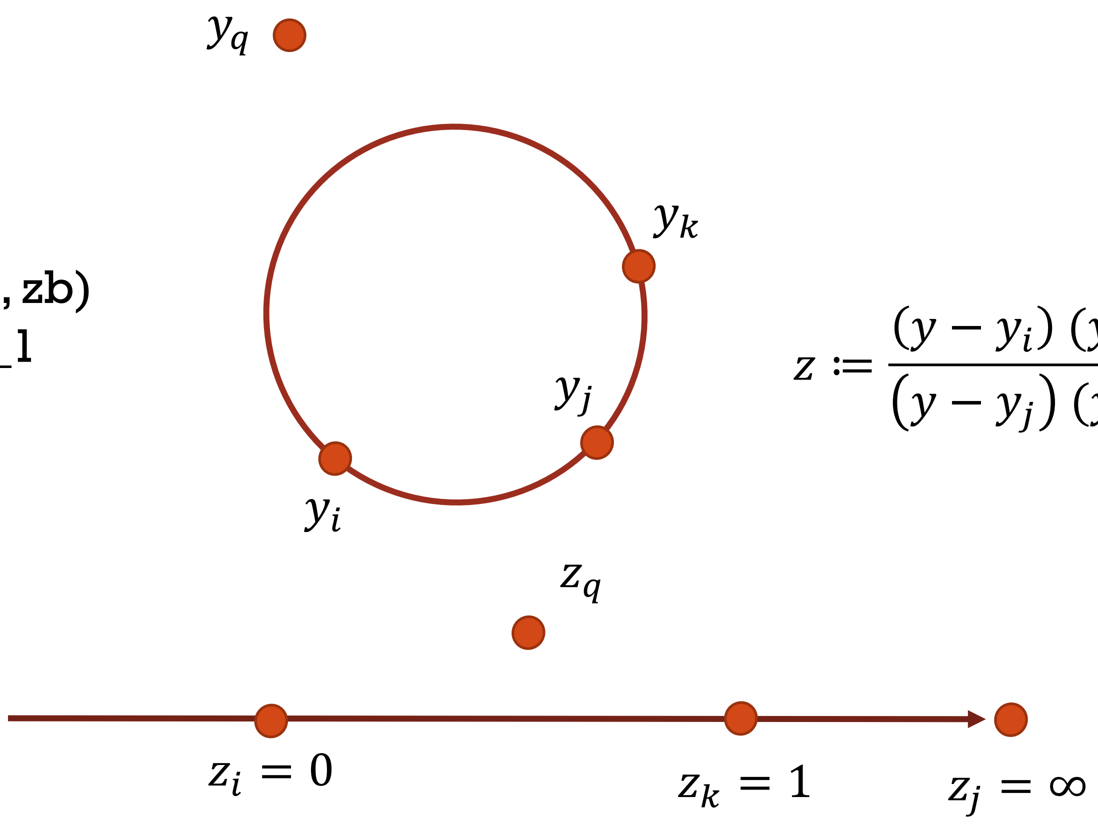
$disc_{A_1} D_i(z)$ are no longer real-analytic,
they develop branch cut on the real axis for |z|>1.

Although the argument of $\ln \frac{1-z}{1-\bar{z}}$
is ambiguous along the branch cut, the value of the specific combination $\ln \frac{1-z}{1-\bar{\bar{\bar{z}}}}$ $(\ln \frac{1-z}{1-\bar{z}} + 2\pi\, i\,)\,(\ln \frac{1-z}{1-\bar{z}} - 2\pi\, i\,)$
vanishes everywhere on the branch cut.

Single-valuedness in A1 region

Given (z, zb) are complex conjugate variables,

there is one-to-one correspondence between (z, zb) and a point in kinematic phases-space in the A_1 region



$$z := \frac{(y - y_i)\,(y_j - y_k)}{(y - y_j)\,(y_i - y_k)}$$

Single-valuedness in A1 region

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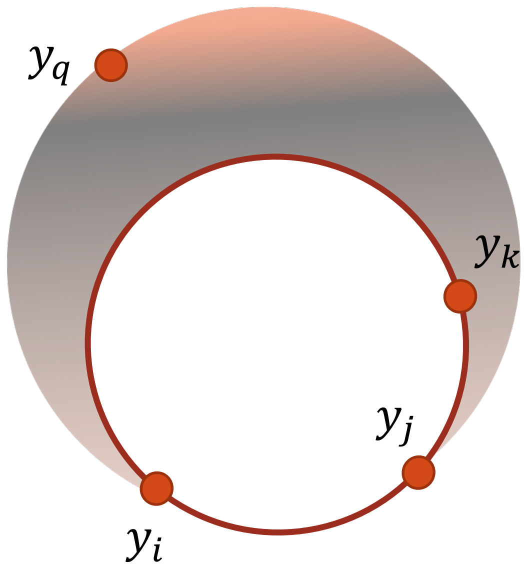
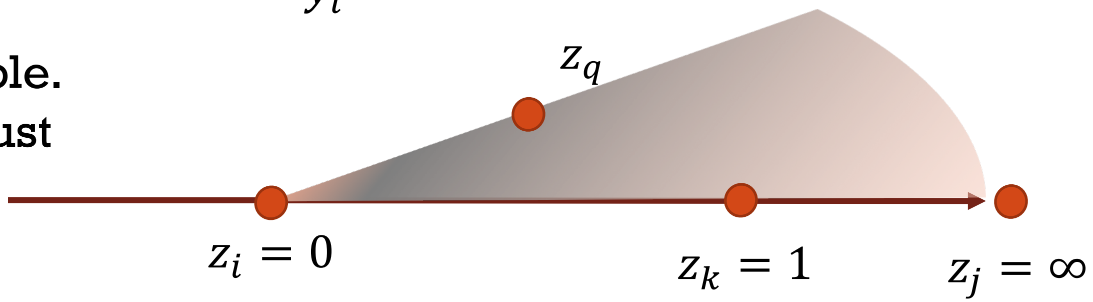


$$z := \frac{(y - y_i)\,(y_j - y_k)}{(y - y_j)\,(y_i - y_k)}$$

The hypersuface z= zb is kinematically accessible. In the vicinity of the boundary, the amplitude must be continuous and ambiguity must cancel.



- $disc_{A_1}D_1(z)$, $disc_{A_1}D_1(1-z)$, $disc_{A_1}D_2(z)$, $disc_{A_1}D_2(1-z)$

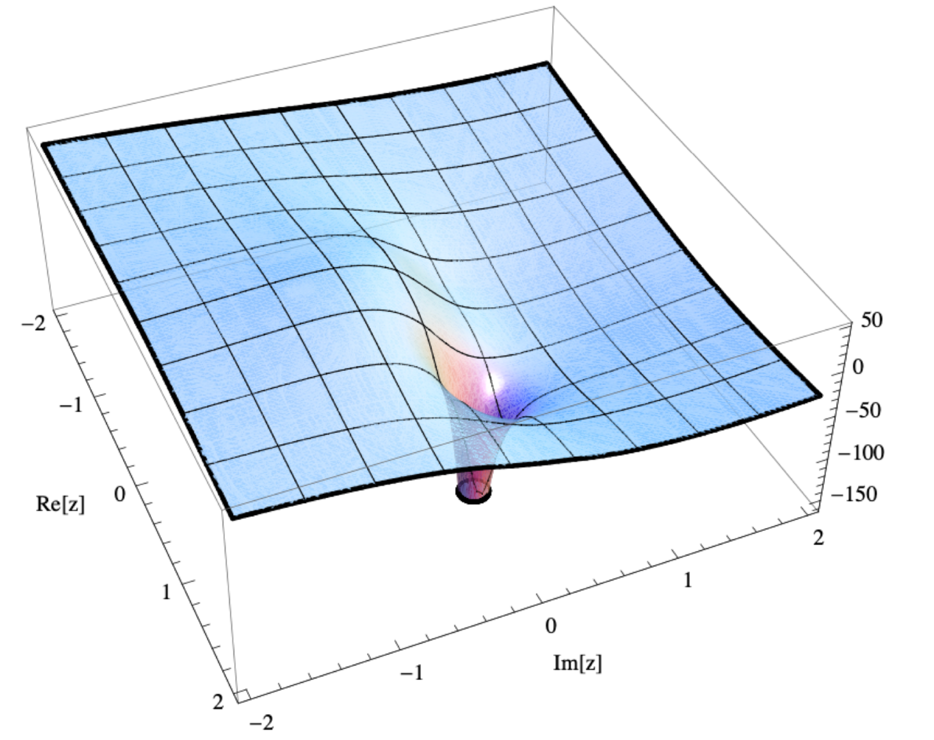
are continuous and differentiable for $\bar{z} = z^*, z \neq 1\ (or\ 0)$

- may construct parity-even functions $disc_{A_1}D_i(z) + disc_{A_1}D_i(\bar{z})$, $\frac{1}{z-\bar{z}}$ [$disc_{A_1}D_i(z)$ - $disc_{A_1}D_i(\bar{z})$] , which have well-defined and non-vanishing limit on the hypersurface z $= \bar{z}$

- These are properties of physical amplitudes, not individual feynman diagrams (in particular, not for F(z, zb))

They offer strong constraints for bootstrapping higher-loop scattering amplitudes

similar property was observed recently in multi-Regge limit of five-point scattering amplitudes Caron-Huot, Chicherin, Henn, Zhang, Zoia [2003.03120]



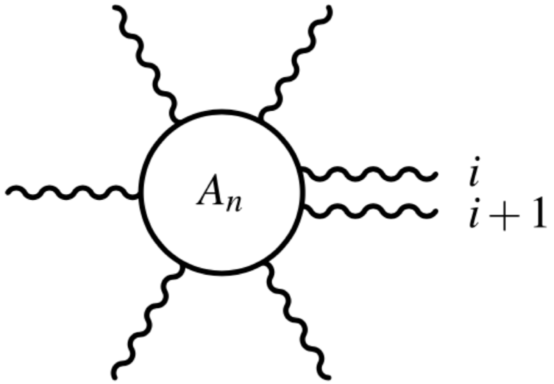
real part $of\ disc_{A_1}D_2(1-z)\ at\ O(\epsilon^0)$

COLLINEAR FACTORIZATION VIOLATION

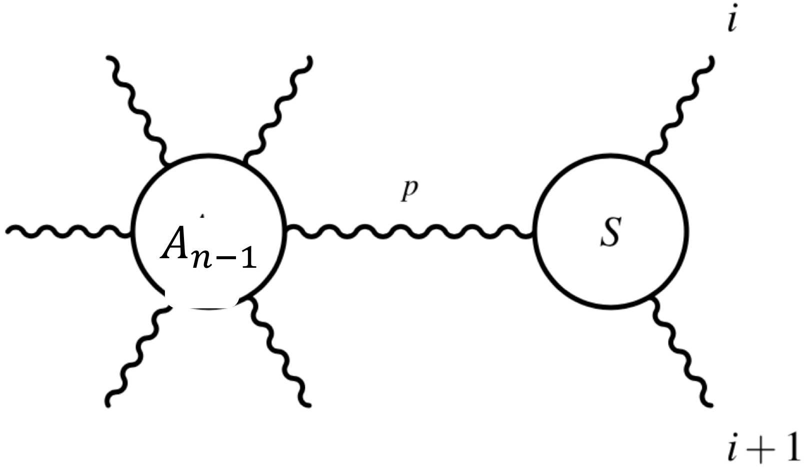
Collinear Factorization

Tree-level amplitudes factorizes on the two-particle pole $P_{i,i+1} = 0$, when two adjacent external momenta are collinear.

$$A_n(\ldots, i, i+1, \ldots) \xrightarrow{i \parallel i+1} \sum_{\lambda} Split_{-\lambda}(z;\ i, i+1)\ A_{n-1}(\ldots, P^{\lambda}, \ldots)$$



$$\lambda_i = \sqrt{z}\,\lambda_P,$$
$$\lambda_{i+1} = \sqrt{1-z}\,\lambda_P$$



Splitting amplitudes are independent of color or kinematics of non-collinear external legs

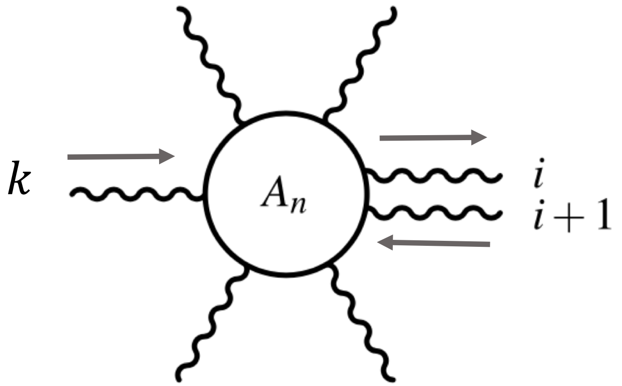
The statement holds to all-loop order for time-like splitting s_{i,i+1} >0 (as a consequence of color coherence). → tripole terms are power-suppressed collinear limit in A0 region

Collinear Factorization violation

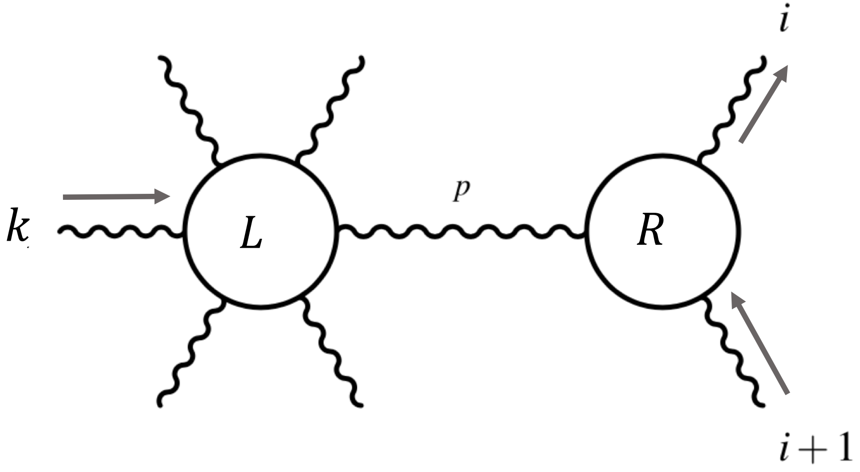
Splitting amplitude depends on color and kinematics of non-collinear external legs

Space-like splitting i+1 → i + P :

$$A_n(\ldots, i, i+1, \ldots) \xrightarrow{i \parallel i+1} \sum_{\lambda} Split_{-\lambda} \;\; A_{n-1}(\ldots, P^{\lambda}, \ldots)$$



$\neq$



The physical origin of the breakdown is related to the feynman prescription(causality of the theory).

$$\frac{1}{\epsilon}\,\gamma_K \sum \log(\frac{-|s_{ij}|e^{-i\pi\lambda_{ij}}}{\mu^2})$$

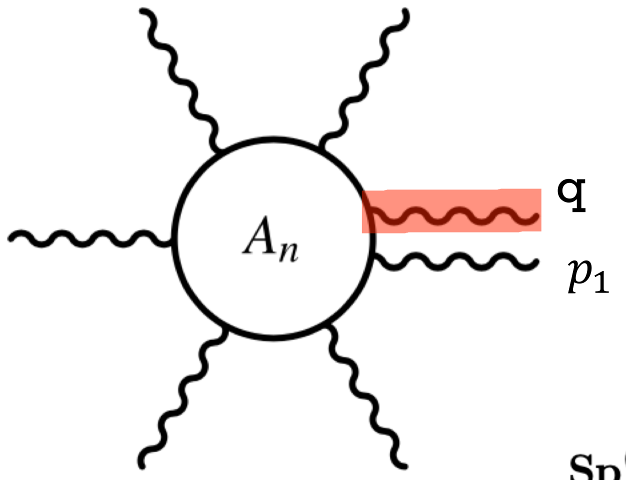
$\lambda_{ij} = 1,$ *both incoming/outgoing,*

$\lambda_{ij} = 0,$ *otherwise.*

$$\frac{i\pi}{\epsilon} \times \sum_{i\in R, j\in L} T_i \cdot T_j \,\lambda_{ij} = \sum_{j\in L} T_P \cdot T_j \,\lambda_{Pj} + \; 2\, T_i \cdot T_k + \text{cnumber}$$

The two-loop splitting amplitudes contain the non-fac. IR poles, π^2/ϵ^2 which distinguish the direction of non-collinear legs [1112.4405] [1206.6363].

Soft-collinear Factorization



Consider L-loop dipole emission, with q collinear to p_1,
where particle 1 is an incoming parton with momentum -p_1

$$S_a^{(L)}\,(q^+,\{p_i\}) \;\to \left(V_{ij}^{q}\right)^{L} f_{abc}T_i^b T_j^c\; C_L(\epsilon)\frac{\langle ij\rangle}{\langle\, iq\rangle\langle qj\,\rangle} \qquad V_{ij}^{q} := \left[\frac{\mu^2\,(-s_{ij})}{(-s_{iq})(-s_{qj})}\right]^{\epsilon}.$$

taking collinear limit, $\sqrt{x_q} \sim \dfrac{\langle k\; q\,\rangle}{\langle\, k\; 1\rangle}\,,\forall\; k \neq 1$

$$\mathbf{Sp}^{(L)}\Big|_{\text{dipole}} \overset{q-\text{soft}}{\simeq} \left(\frac{\mu^2}{x_q s_{1q}}\right)^{L\epsilon} C_L(\epsilon)\left\{\mathrm{i}\sin(L\pi\epsilon)\sum_{k\neq 1}(-1)^{\lambda_{kq}}\boldsymbol{T}_q\cdot\boldsymbol{T}_k + \frac{C_A}{2}\cos(L\pi\epsilon)\right\}\mathbf{Sp}^{(0)}$$

$$Sp_{-}^{(0)}\left(z_q, q^+, 1^+\,\right) = \;-T_1\frac{1}{\sqrt{x_q}}\frac{1}{\langle\, q1\rangle}$$

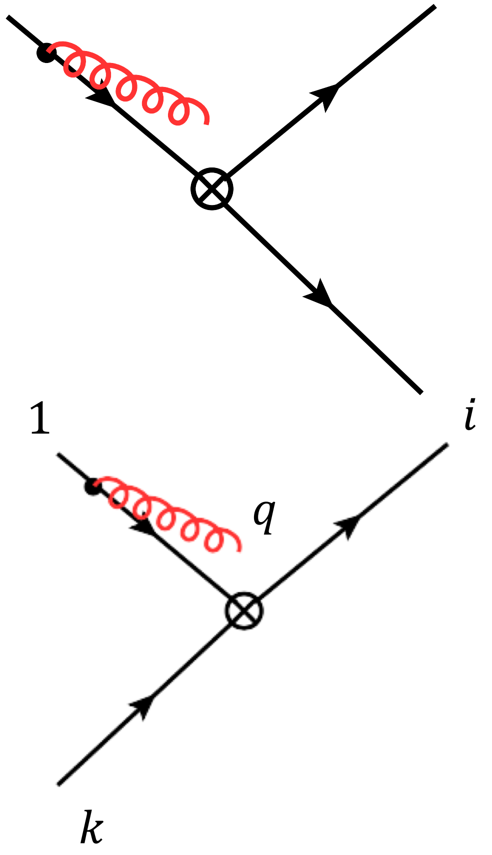
Factorization breaking terms in th dipole formular are purely imaginary (anti-hermittian), do not account for the non-universal IR pole in the soft limit for the splitting amplitude.

Origin of collinear factorization violation

Consider the tripole terms in the space-like collinear limit:

--in A_2 region: suppressed in the collinear limit

-- in A_1 region where {j(=1) , k} are incoming and {i, q} are outgoing: do not vanish in the collinear limit, due to the A1-discontinuity.



Region	Kinematics	analytic continuation rule	
A_1	j,k incoming, q,i outgoing	$u_k^{ij} \to \lvert u_k^{ij} \rvert$	$v_k^{ij} \to \lvert v_k^{ij} \rvert e^{-2i\pi}$
A_2	i incoming, q,j,k outgoing	$u_k^{ij} \to \lvert u_k^{ij} \rvert$	$v_k^{ij} \to \lvert v_k^{ij} \rvert$

$$\lim_{z,\bar{z}\to 1} \left[S^{+}_{a,\{i,j,k\}} \Big|_{A_1} \right] = \lim_{z,\bar{z}\to 1} disc_{A_1} S^{+}_{a,\{i,j,k\}}$$

$$= T_1^{a_1} \frac{1}{\sqrt{-x_q}\,\langle 1q\rangle} \left(\frac{\mu^2}{x_q s_{1q}} \right)^{2\epsilon} \exp[-2i\pi] \;\; 2\, T_i^{a_i} T_k^{a_k}$$

$$\times \lim_{z,\bar{z}\to 1} \left[f^{a a_i b} f^{b a_1 a_k} \, disc_{A_1} D_1 \,(1-z, 1-\bar{z}) + f^{a a_1 b} f^{b a_k a_i} \, disc_{A_1} D_2 \,(1-z, 1-\bar{z}) \right]$$

Two-loop space-like splitting amplitude in the soft-collinear limit

$$\mathbf{Sp}^{(2)}\Big|_{\text{tripole}} \overset{q-\text{soft}}{\simeq} -\frac{1}{4} \sum_{\substack{\text{tripoles}\\ \{i,1,k\}}} \boldsymbol{S}^{+,(2)}_{a,\{i,1,k\}}\Big|_{q\|p_1}$$

$$= \left(\frac{\mu^2}{x_q s_{1q}} \right)^{2\epsilon} \sum_{i\neq k\neq 1} \delta_{0,\lambda_{ik}} \delta_{1,\lambda_{1k}} \Bigg\{ f^{b a_k a_i} \boldsymbol{T}_q^b \boldsymbol{T}_k^{a_k} \boldsymbol{T}_i^{a_i} \times \Bigg[\qquad (4.27)$$

$$\frac{1}{\epsilon^2}\left(\mathrm{i}\pi \log v_k^{1i} - \pi^2\right) - \frac{\mathrm{i}\pi^3}{3} \log v_k^{1i} + 4\mathrm{i}\pi\zeta_3 + 30\zeta_4 + \frac{8\pi}{3}\left(\arg(z_k^{1i})^3 - \pi^2 \arg(z_k^{1i})\right)\Bigg]$$

$$+ \left[(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)(\boldsymbol{T}_q\cdot\boldsymbol{T}_k) + (\boldsymbol{T}_q\cdot\boldsymbol{T}_k)(\boldsymbol{T}_q\cdot\boldsymbol{T}_i) \right] \left(\frac{\pi^2}{\epsilon^2} - 30\zeta_4 \right) \Bigg\} \mathbf{Sp}^{(0)},$$

The factorization breaking IR poles agrees with litereature [1112.4405] [1206.6363].

Squared splitting amplitude

$$\mathbf{Sp}^\dagger\mathbf{Sp}\Big|_{\text{non-fac.}} \overset{q-\text{soft}}{\simeq} \bar{a}^2 g_s^2 \sum_{i\neq k\neq 1} \delta_{0,\lambda_{ik}} \mathbf{Sp}^{(0)\dagger} \Bigg\{ \Big[(\boldsymbol{T}_q\cdot\boldsymbol{T}_i)\,(\boldsymbol{T}_q\cdot\boldsymbol{T}_k) + (\boldsymbol{T}_q\cdot\boldsymbol{T}_k)\,(\boldsymbol{T}_q\cdot\boldsymbol{T}_i) \Big] (-15\,\zeta_4)$$

$$+\, 2\pi \mathrm{i}\, \delta_{1,\lambda_{1k}}\, f^{b a_k a_i} \boldsymbol{T}_q^b \boldsymbol{T}_k^{a_k} \boldsymbol{T}_i^{a_i} \left(\frac{\mu^2}{x_q s_{1q}}\right)^{2\epsilon} \left[\left(\frac{1}{\epsilon^2} - 2\zeta_2\right) \log v_k^{1i} + 4\zeta_3\right] \Bigg\} \mathbf{Sp}^{(0)} + \mathcal{O}(\bar{a}^4)\,.$$

The second line us given by commutator between two Hermitian operator [(T q · T i), (T q · T k)]. At N^3LO, expectation value on tree amplitudes ⟨M(0) | ··· | M(0)⟩ is traceless in color space, the color sum vanishes.
The second line will contribute only at N^4LO and beyond.

$$v_k^{1i} = \frac{s_{ik}s_{1q}}{s_{1i}s_{kq}}\,, \qquad z_k^{1i} = \frac{\langle ki\rangle\langle 1q\rangle}{\langle 1i\rangle\langle kq\rangle}\,.$$

Factorization violation comes from the first line:

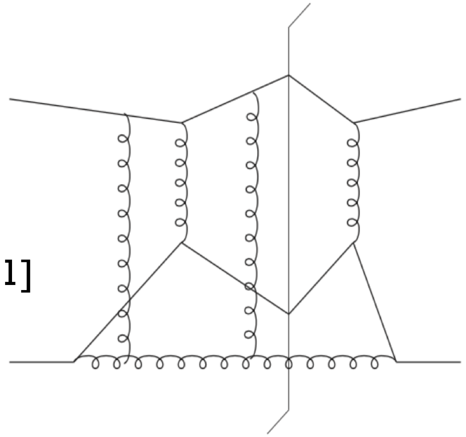
The non-fac. IR poles cancel at cross section level up to N^3LO
We made a concrete argument that the finite part does not factorize.

Mechanism for factorization breaking has been studied in various contexts:

- Transverse-momentum-dependent pdf factorization

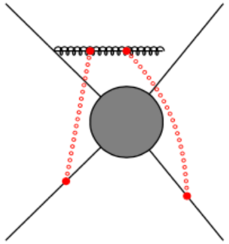
  An counterexample was construct for the single-spin asymmetry (in a simplified model theory)

  .



Collins,Qiu, [0705.2141]

- Event shapes at hadron colliders



$$
\vdots = -(\mathbf{T}_2 \cdot \mathbf{T}_j)(\mathbf{T}_2 \cdot \mathbf{T}_3)\,\mathbf{Sp}^0\,\overline{\mathcal{M}}^0 \times \left(\frac{\alpha_s}{2\pi}\right)^2 (i\pi)^2 \left(\frac{4\pi\mu^2}{\vec{p}_{2,\perp}^{\,2}}\right)^{2\epsilon} [\Gamma(-\epsilon)]^2 \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}
$$

In an EFT for Glauber gluon, a particular type of effective diagram produces the same two-loop constant as we find the soft emission factor.

Schwartz, K.Y., Zhu [1703.08572]

We see a convergence of stories in different frameworks.

New type of phase-space collinear singularity

Consider space-like collinear splitting: $P_1 \to (1-x_q)P_1 + x_q P_1$

Phase-space integrals of the 1-> 2 splitting amplitude generate collinear divergences that depend on the color of non-collinear particles

$$\int d^2q_T\, |Sp|^2{}_{non-fac.} := \int \frac{d^2q_T}{q_T^2} P_{non-fac.}(1-x_q)$$

Given the two-loop result for $\lim_{x_q \to 0} |Sp|^{1\to 2}$

$$\lim_{x_q\to 0} P_{non-fac.}(1-x_q) = a^3 \sum_{outgoing\ j} \left(T_1[(T_q\cdot T_2)(T_q\cdot T_j) + (T_q\cdot T_j)(T_q\cdot T_2)]T_1\right)(-15\,\zeta_4) \quad +O(a^4)$$

Relavant at N^3LO for partonic cross-section for 1+2 → q+ 3+ 4 + … with high-pT jets in the final state (e.g. Dijet production at hadron colliders)

understanding multi-parton color evolution in the long distance is crucial for the estimation of theoretical uncertainties.

Conventional picture of factorization of hadronic cross section:

$$d\sigma = \int \frac{d\xi_A}{\xi_A}\frac{d\xi_B}{\xi_B}\ \phi_{\frac{a}{A}}(\xi_A, \mu_f)\ d\widehat{\sigma_{ab}}(\frac{x_A}{\xi_A}, Q, \mu_f)\phi_{\frac{b}{B}}(\xi_B, \mu_f) + O(\Lambda_{QCD}/Q)$$

Factorization scale dependence of $\mathrm{d}\hat{\sigma}$ is process-independent, compensated by pdf evolution

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\phi_{i/h}(x,\mu,\mu^2) = \sum_{j=f,\bar{f},G}\int_x^1 \frac{\mathrm{d}\xi}{\xi}P_{ij}\left(\frac{x}{\xi}, \alpha_s(\mu^2)\right)\phi_{j/h}(\xi,\mu,\mu^2)$$

[Gribov, Lipatov, 1972a];
[Altarelli, Parisi ,1977]

Pdf evolution kernel at N^3LO and beyond might need to be corrected by $P_{non-fac}(x)$ depending on the specific underlying scattering process.

$$\lim_{x\to 1} P_{non-fac.}(x) \neq^{?} 0 \qquad \qquad for\ multi-jet\ production\ at\ the\ LHC$$

Need to compute the phase-space integral over one-loop $|Sp|^{1\to 3}$ to confirm this argument!

Summary

We provide the first result for two-loop soft emission factor beyond leading color. The result reveals certain intricate analytic properties of multi-parton scattering amplitudes and may serve as a building block for studying singularities for N3LO phase-space integrals.

Future directions

- Beyond two loop: bootstrapping higher-loop results from the constraints on their analytic behaviours
- Application to precision event shapes at hadron colliders, where N^3LO is within reach, e.g transverse thrust, transverse energy correlators
- Probing collinear factorization breaking from the soft limit:

need triple-real, one-loop double-real and two-loop single-real soft emission facots (all available)

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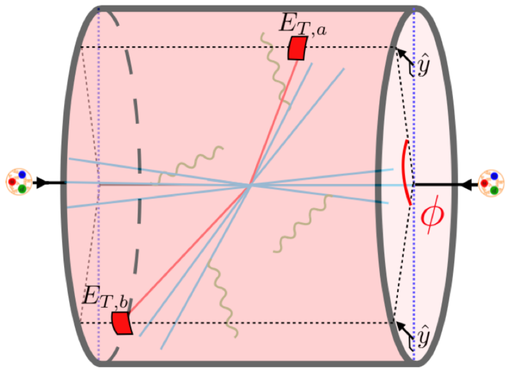

Thank you for your attention .