Next-to-leading non-global logarithms from jet calculus

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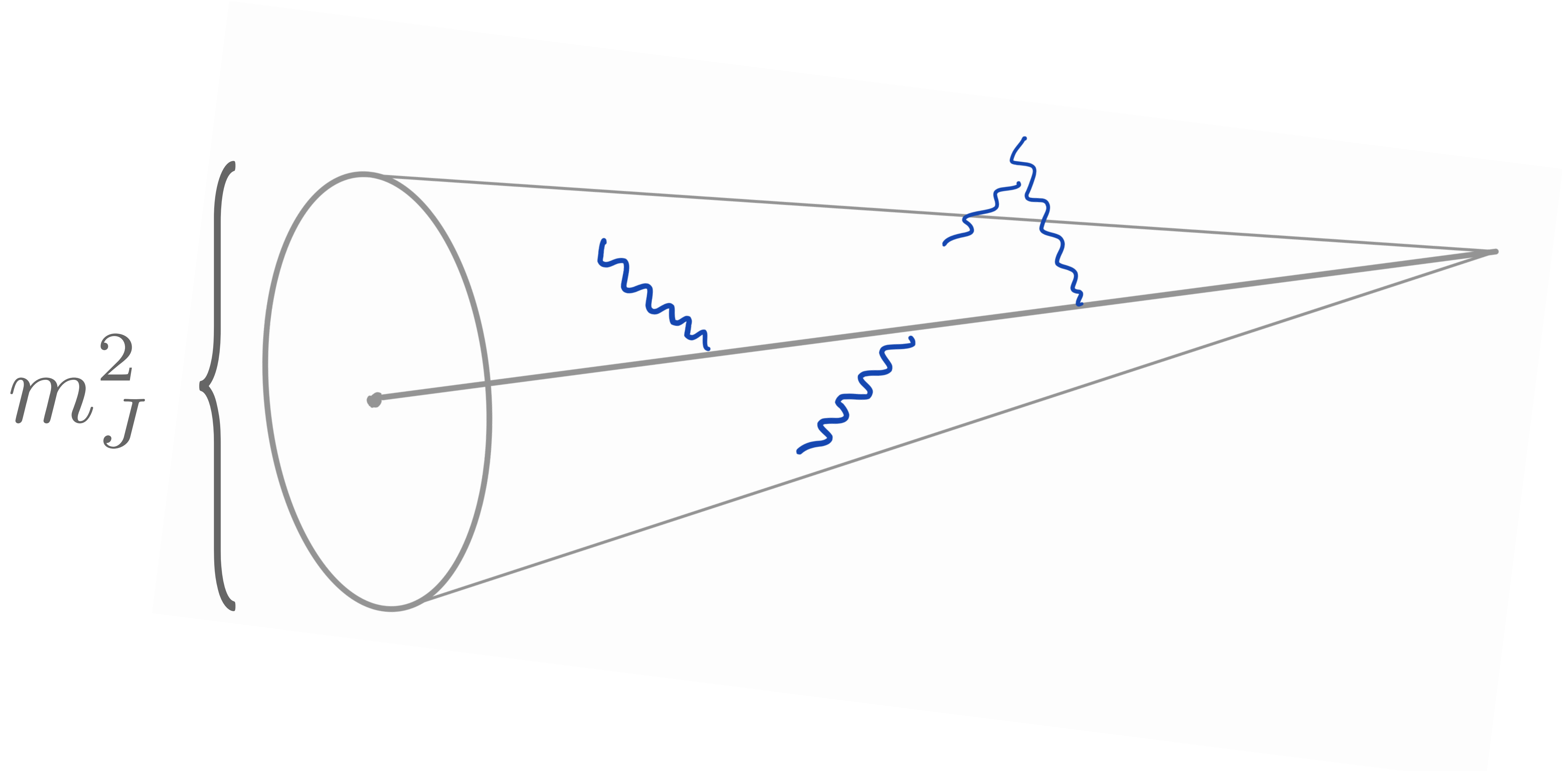
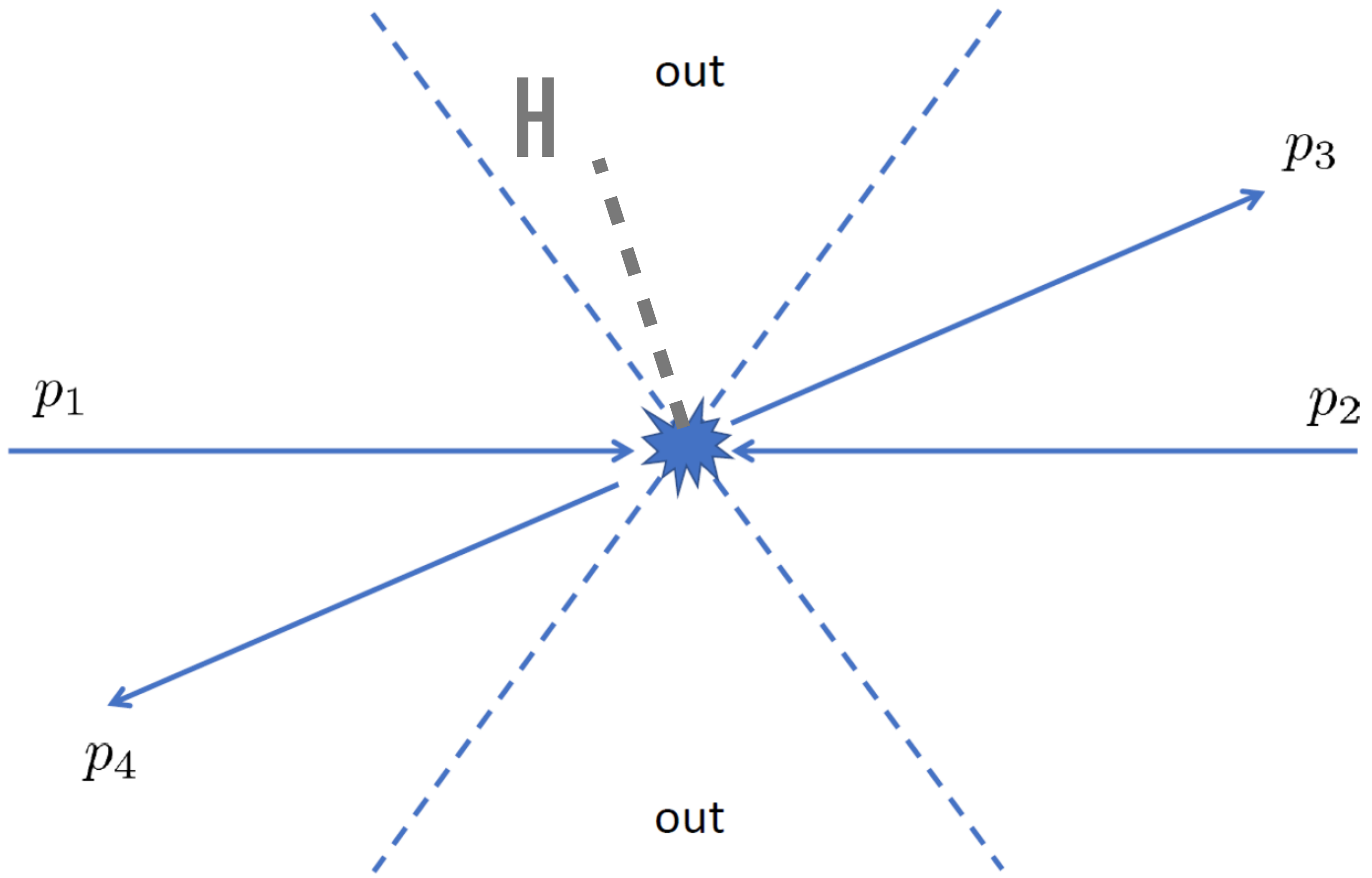
JHEP 2021, 6 (2021) [2104.06416] + [2111.02413]

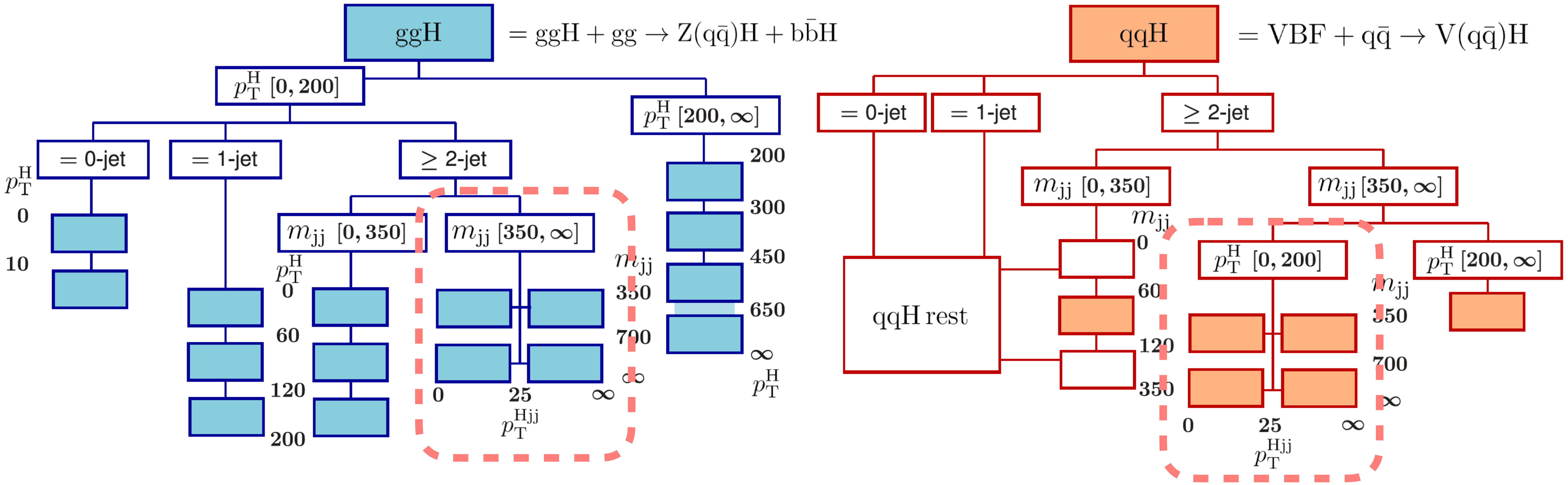
UZH/ETH Theoretical Particle Physics Seminar - November 2021

Non-global logarithms

- Ubiquitous in collider observables
  - use of jets and experimental fiducial cuts (e.g. jet mass, rapidity cuts, isolation, …)
  - reduction of background reactions (e.g. Higgs production in ggF vs. VBF)







[CMS STXS in H→γγ ’21]

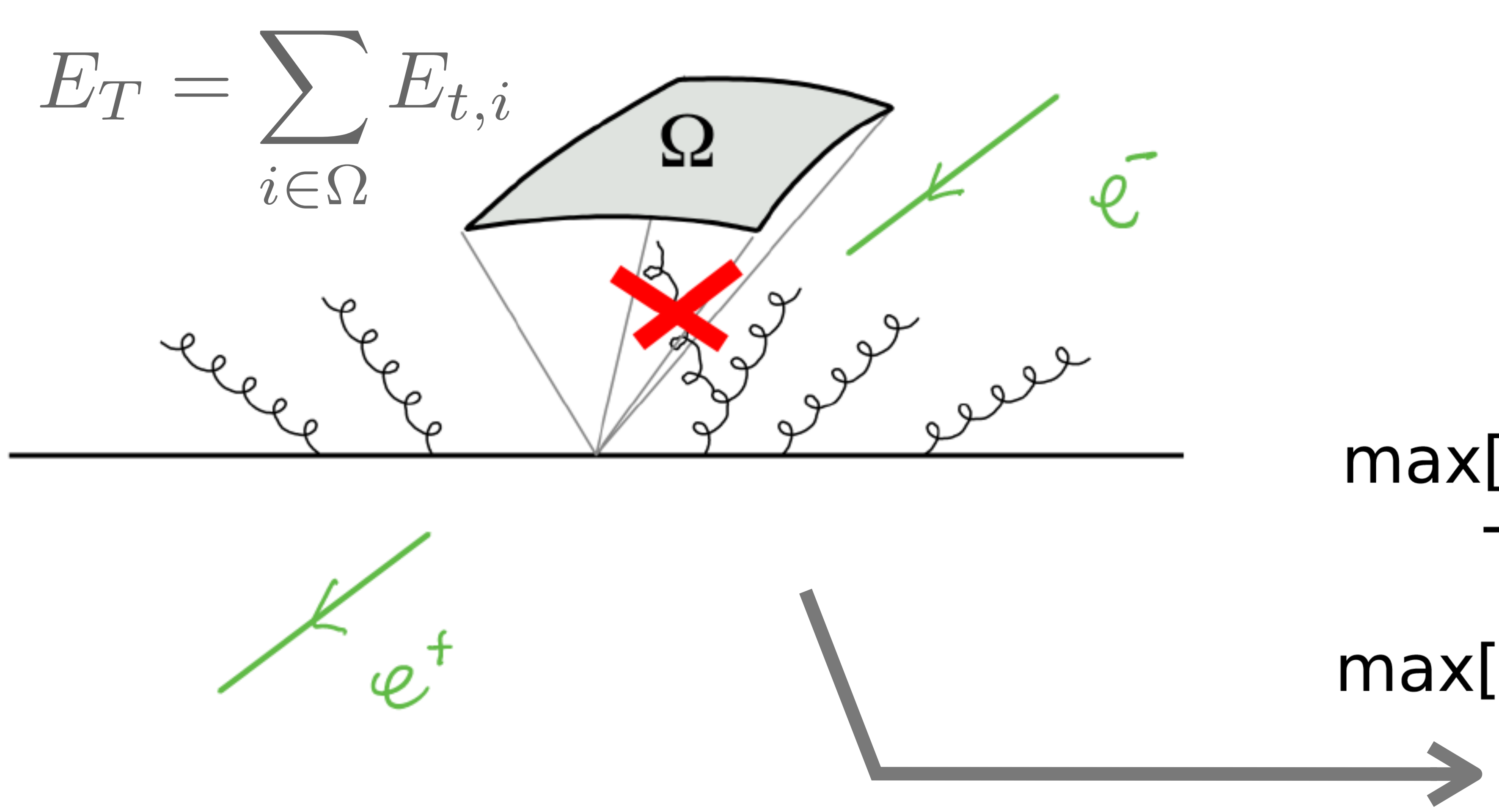
Non-global logarithms

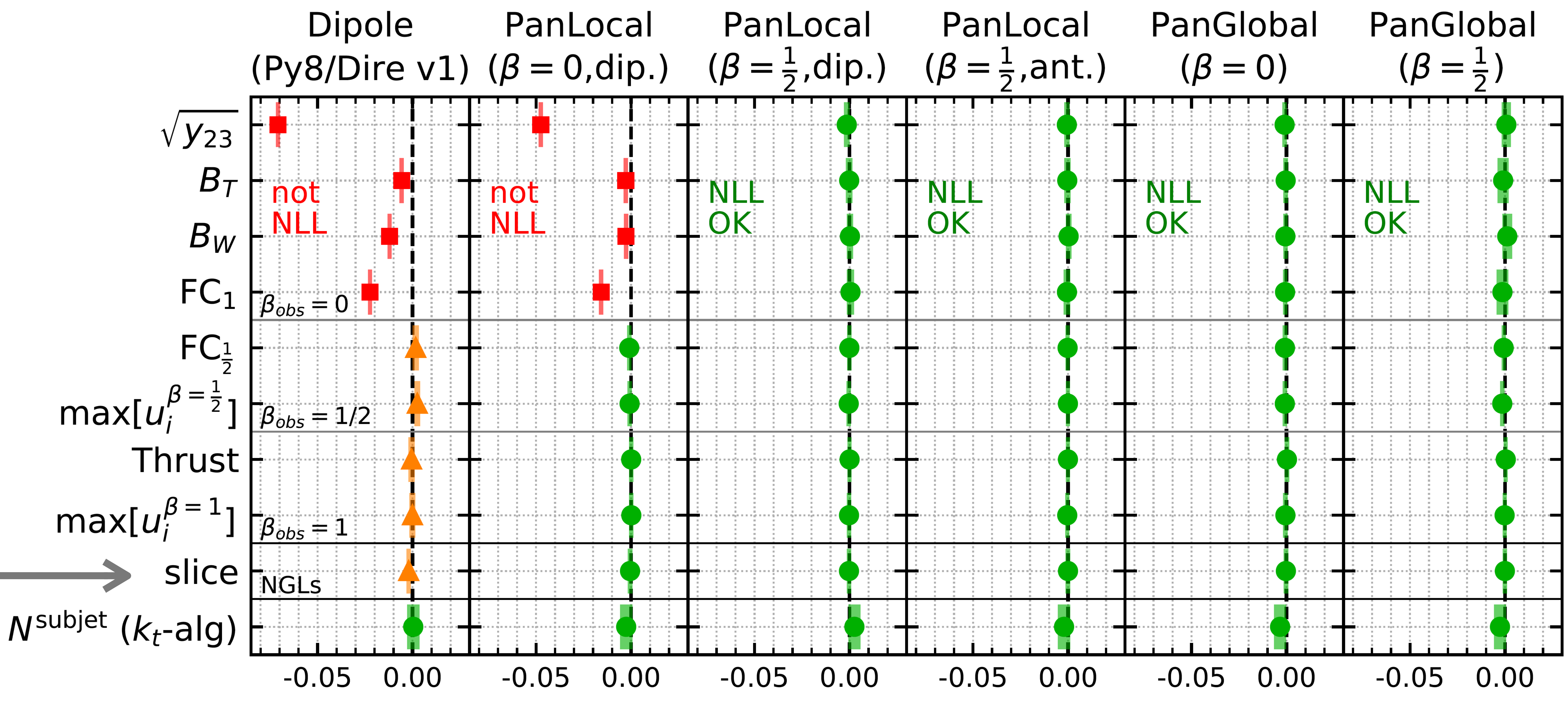
Plot: relative deviation from exact NLL
[in $\alpha_s \to 0$ limit at fixed $\alpha_s L$]

- Their resummation is an essential ingredient for parton showers
  - dipole showers needed to describe them @ LL* (failure of angular ordering designs)

    [Banfi, Corcella, Dasgupta '06]
  - NGLs @ NLL are a building block for NNLL algorithms

    [Dasgupta, Dreyer, Hamilton, PM, Salam, Soyez '20]





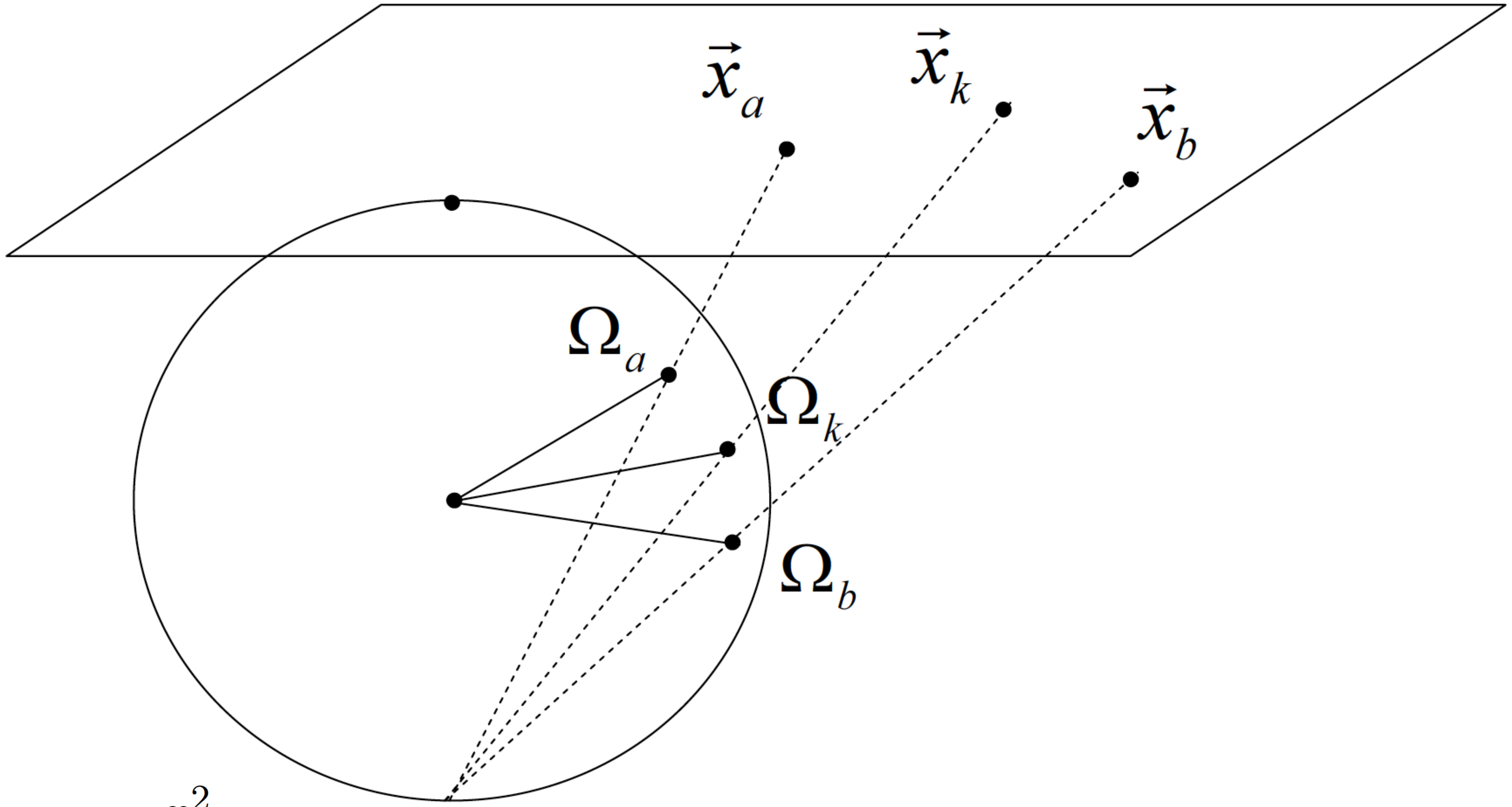
$$\Sigma(\alpha_s, \alpha_s L) = \exp\left[\alpha_s^{-1} g_1(\alpha_s L) + g_2(\alpha_s L) + \mathcal{O}\left(\alpha_s^n L^{n-1}\right)\right]$$

* NGLs start at NLL (i.e. $(\alpha_s L)^n$) in the general case

Non-global logarithms

- Stereographic projection relates NGL evolution equation (BMS) to saturation dynamics in high-energy forward scattering (BK/JIMWLK) at all orders

[Weigert '03; Hatta '08; Caron-Huot '15]



$$\cos\theta = \frac{1-|\vec{x}|^2}{1+|\vec{x}|^2}, \quad \sin\theta = \frac{2|\vec{x}|}{1+|\vec{x}|^2}, \quad \cos\phi = \frac{x^1}{|\vec{x}|}, \quad \sin\phi = \frac{x^2}{|\vec{x}|}$$

$$\frac{d^2\Omega_k}{4\pi}\frac{1-\cos\theta_{ab}}{(1-\cos\theta_{ak})(1-\cos\theta_{bk})} = \frac{d^2\vec{x}_k}{2\pi}\frac{(\vec{x}_{ab})^2}{(\vec{x}_{ak})^2(\vec{x}_{bk})^2}$$

i.e. distribution of small-x gluons in the transverse plane is equivalent to angular distribution of soft gluons on the sphere at infinity

Non-global logarithms

- Resummation of LL corrections known for a long time and studied in depth

[Dasgupta, Salam ’01-’02; Banfi, Marchesini, Smye ’02]
[Forshaw, Kyrieleis, Seymour ’06; Forshaw, Keates, Marzani ’09]
Full Nc in: [Weigert ’03; Hatta, Ueda ’13-‘20 (+Hagiwara ’15)] …

- Revived interest more recently and new formulations with modern QFT techniques

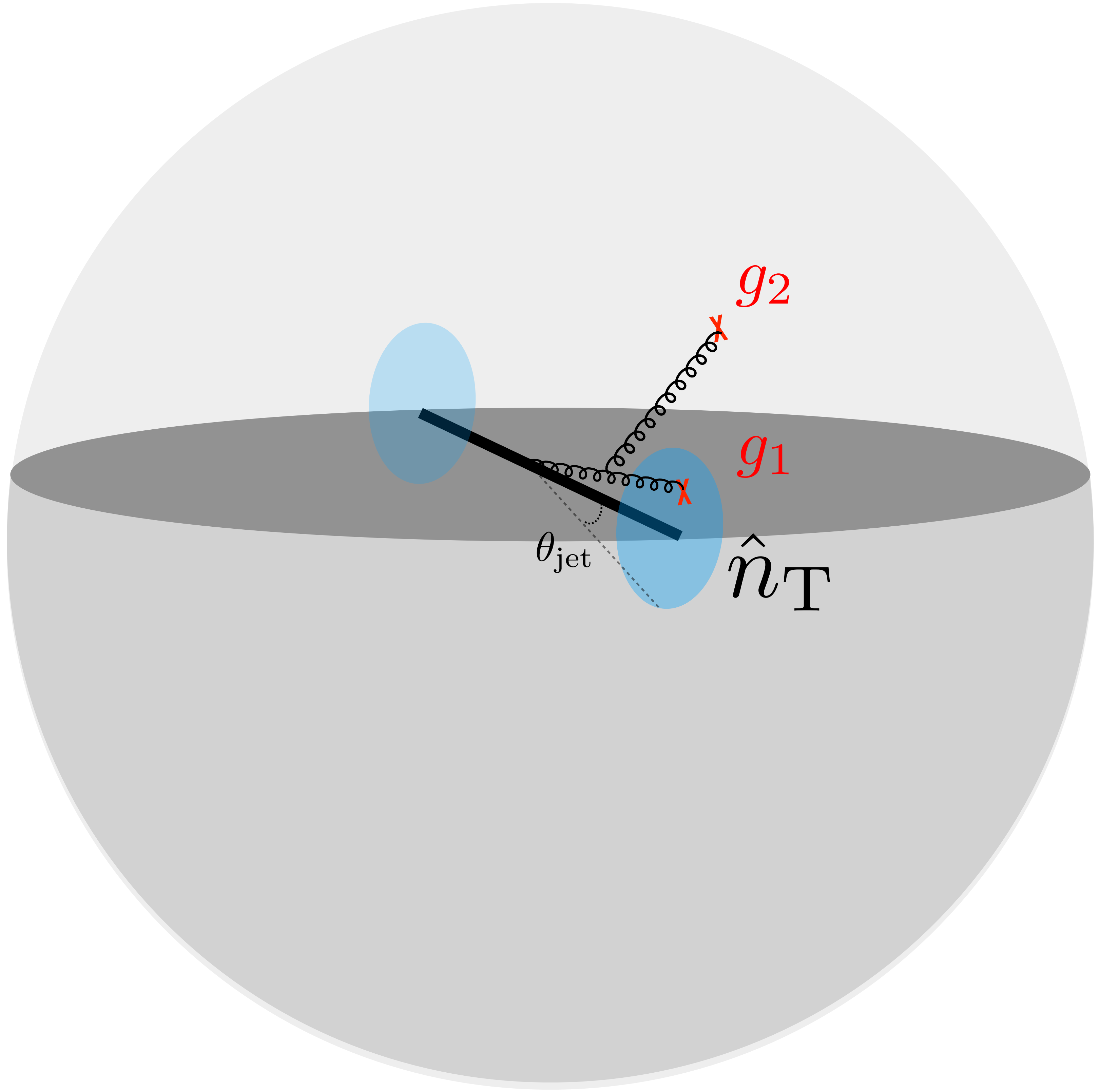
e.g. [Becher, Neubert, Rothen, Shao ’15-’16 (+ Pecjak ’16, Rahn ’17, Balsiger ’18-’19, Ferroglia ’20); Larkoski, Moult, Neill ’15-’16; Caron-Huot ’16; Angeles Martinez, De Angelis, Forshaw, Plaetzer, Seymour ’18] …

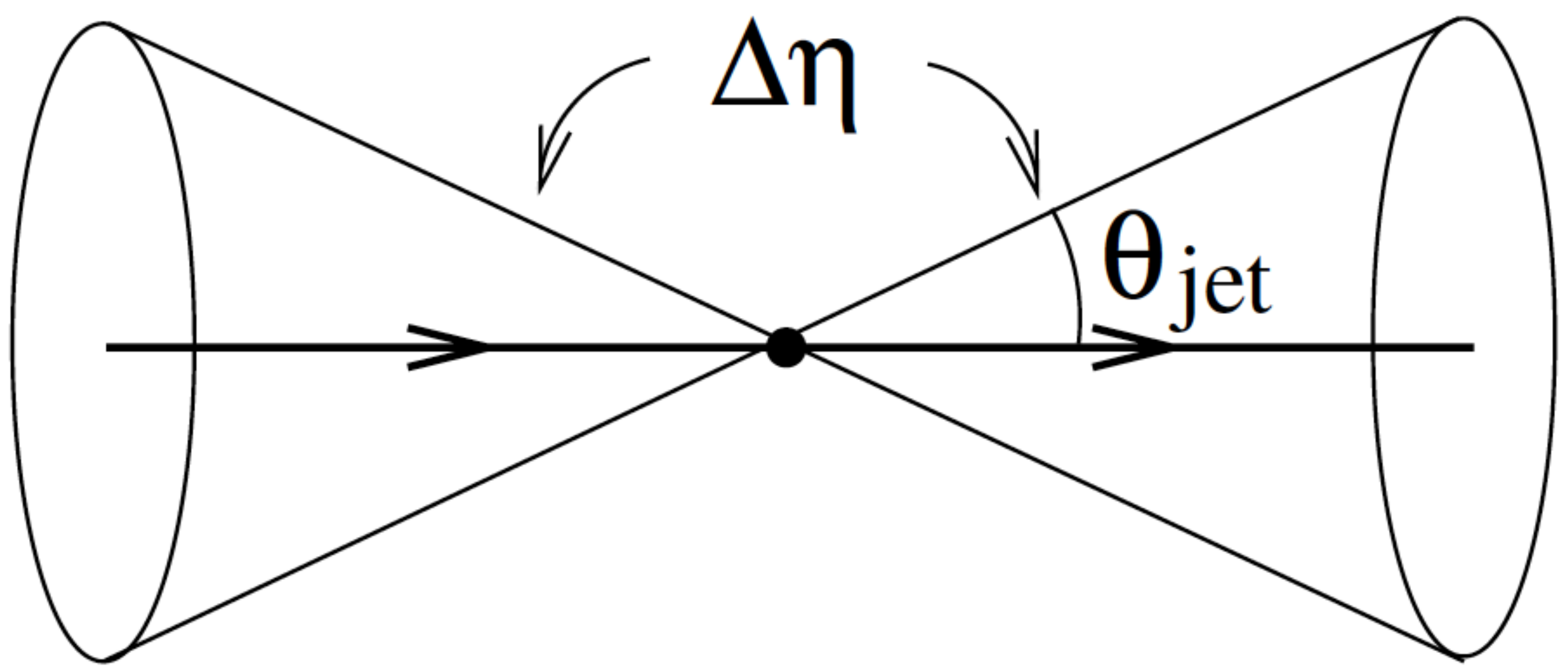
- Resummation of NLL corrections remains a great technical challenge due to the complexity of the geometry and colour structure of a typical NG problem

- GOAL of this work ⇒ formulate the problem (in planar limit) in such a way that can be integrated numerically for a variety of observables & processes at once

A toy model: cone-jet cross section with a veto

- A simple laboratory to study these radiative corrections is the production of two cone jets at lepton colliders, with a veto on radiation in the interjet region





$$\Delta\eta := \ln\frac{1+c}{1-c}, \qquad c = \cos\theta_{\rm jet}$$

Apply a veto e.g. on energy or transverse energy (scalar p_t sum) of the radiation in the gap.
Need to calculate distribution of soft gluons on the sphere as a function of the veto scale

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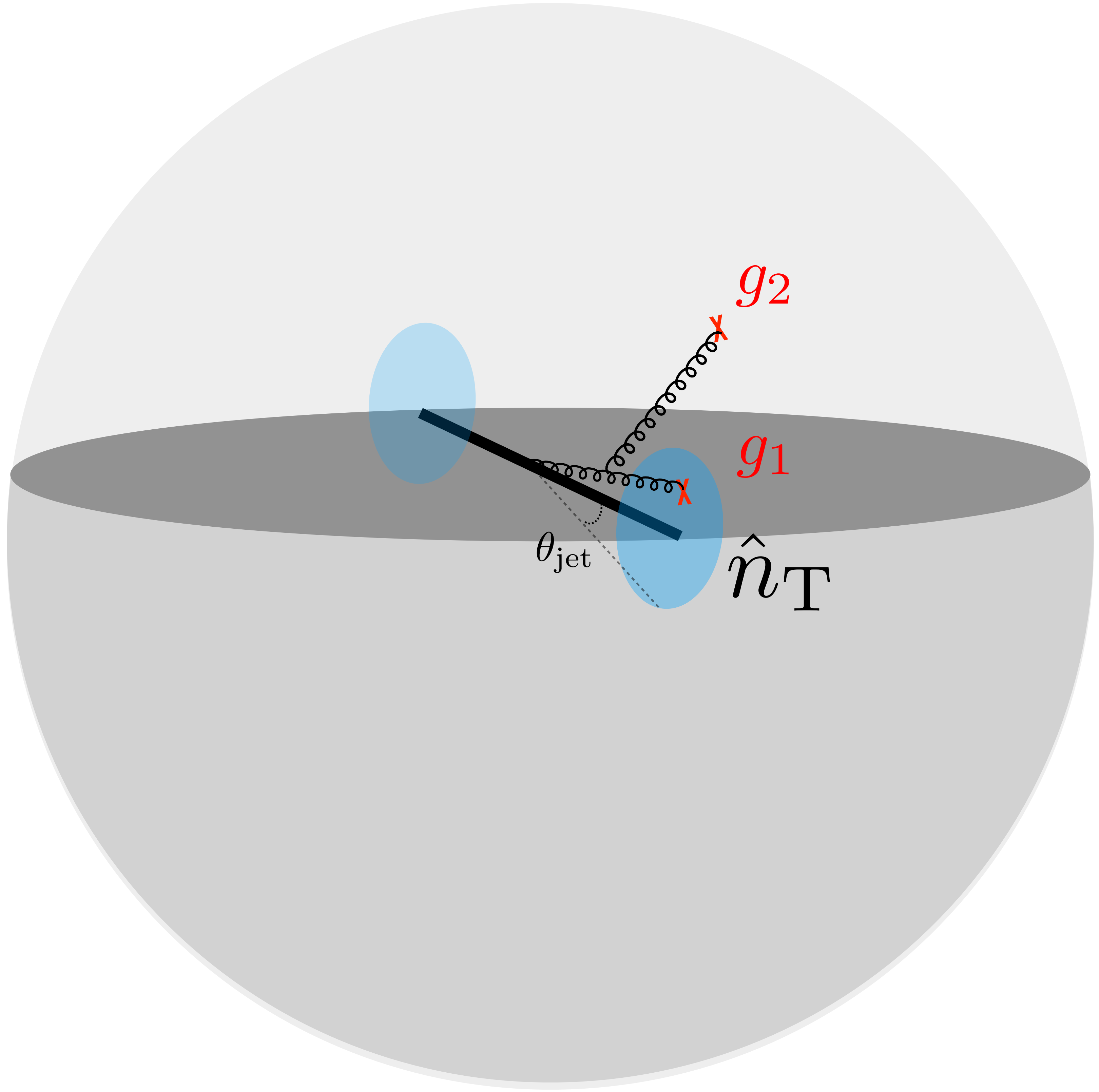

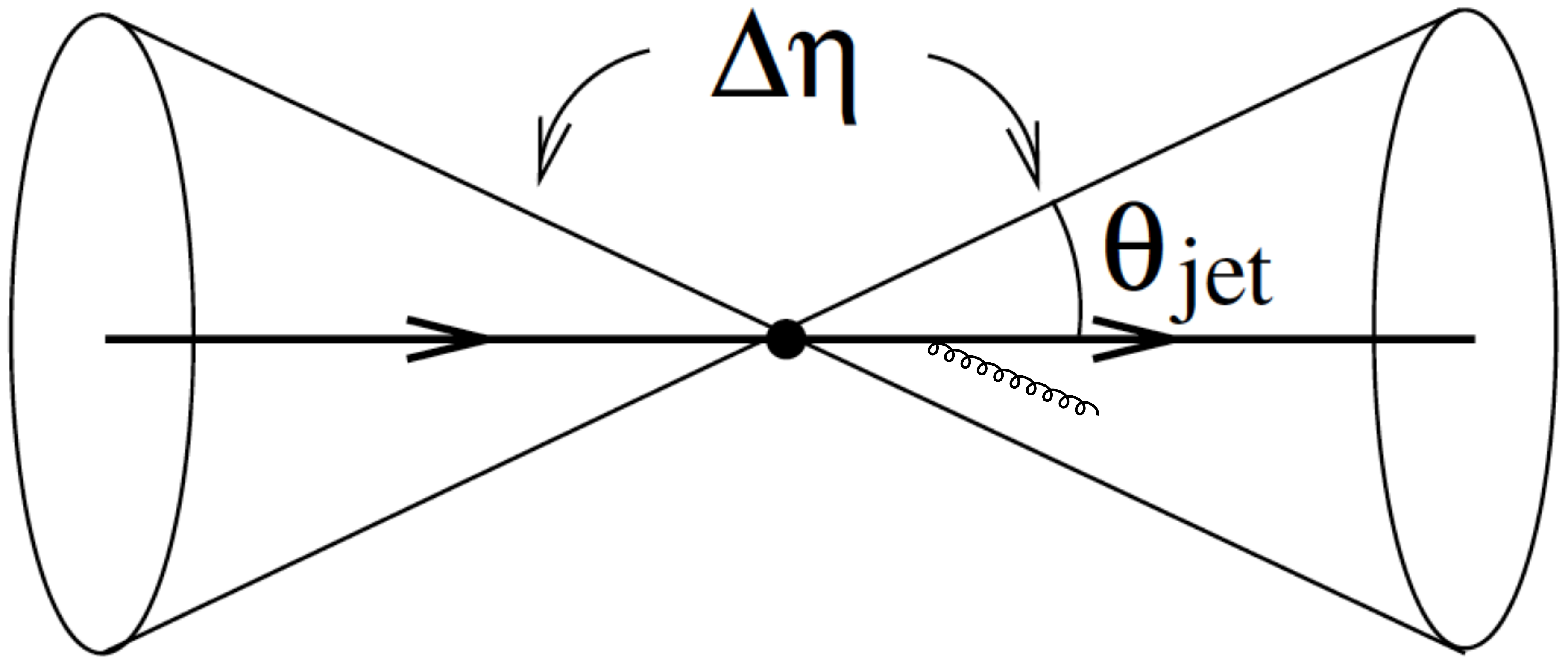

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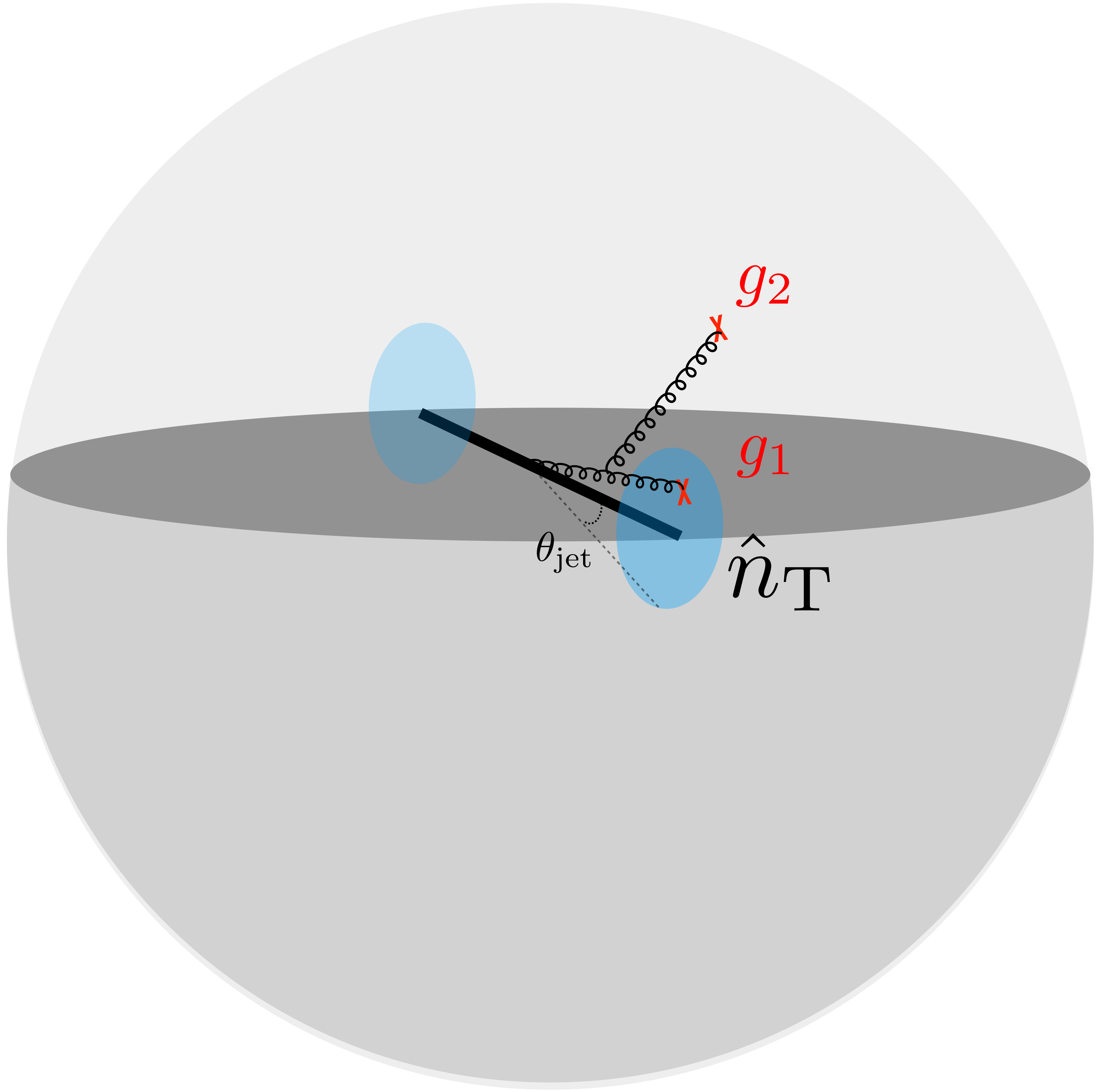

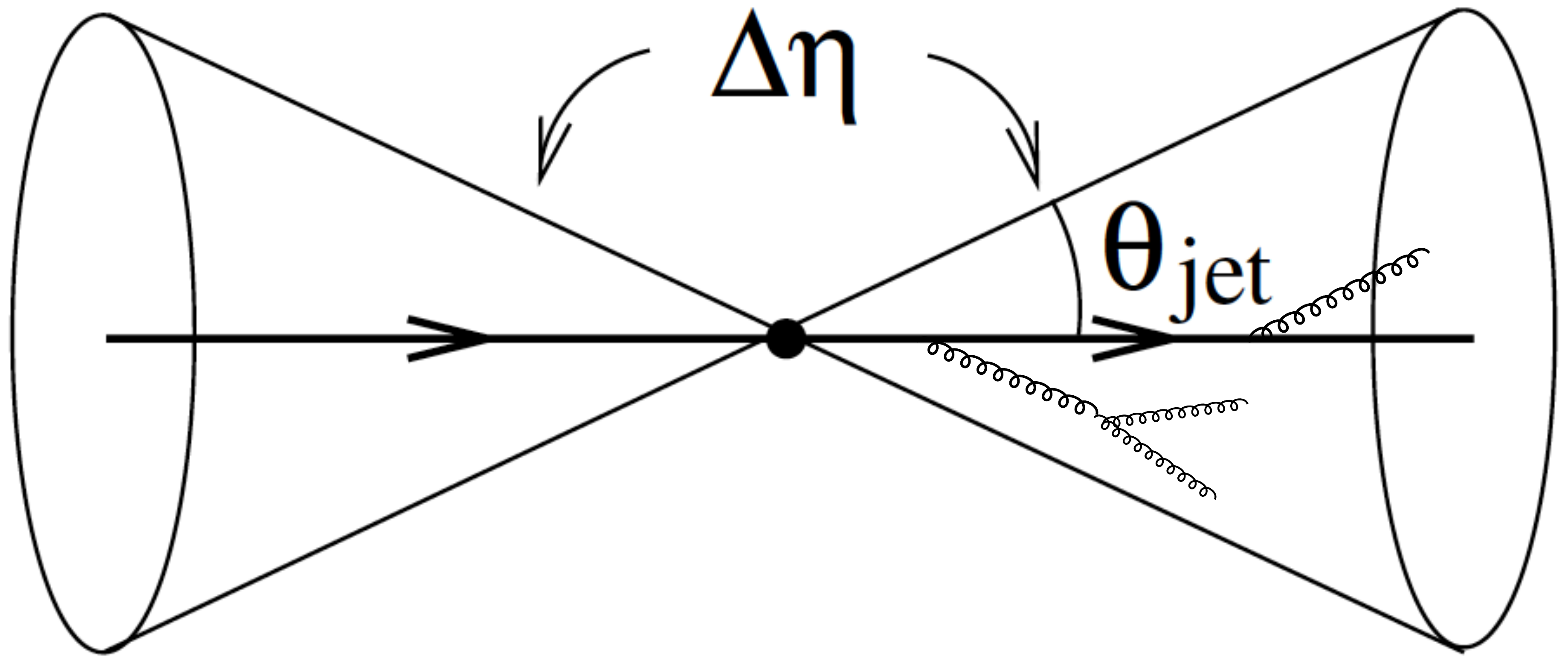

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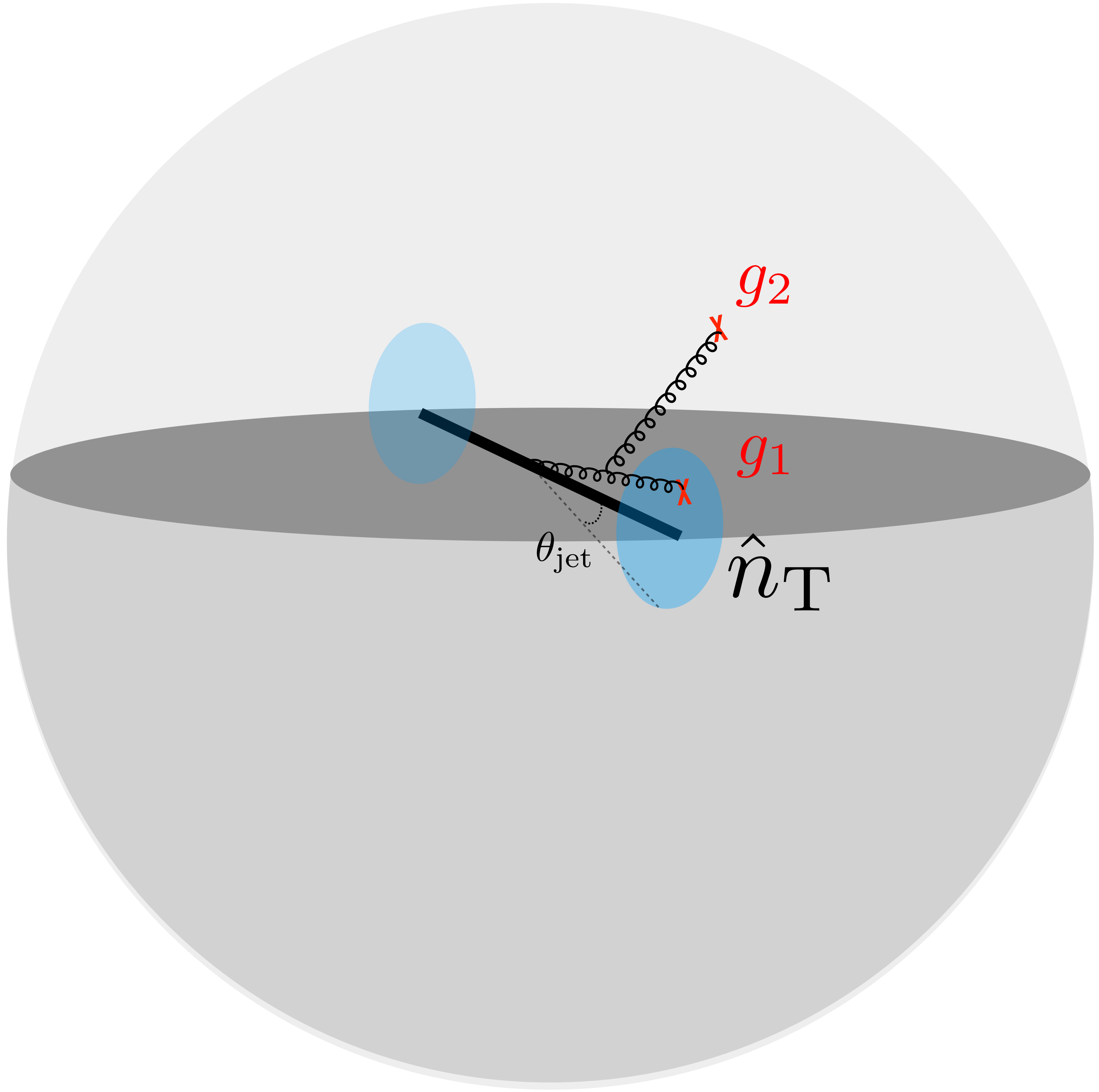

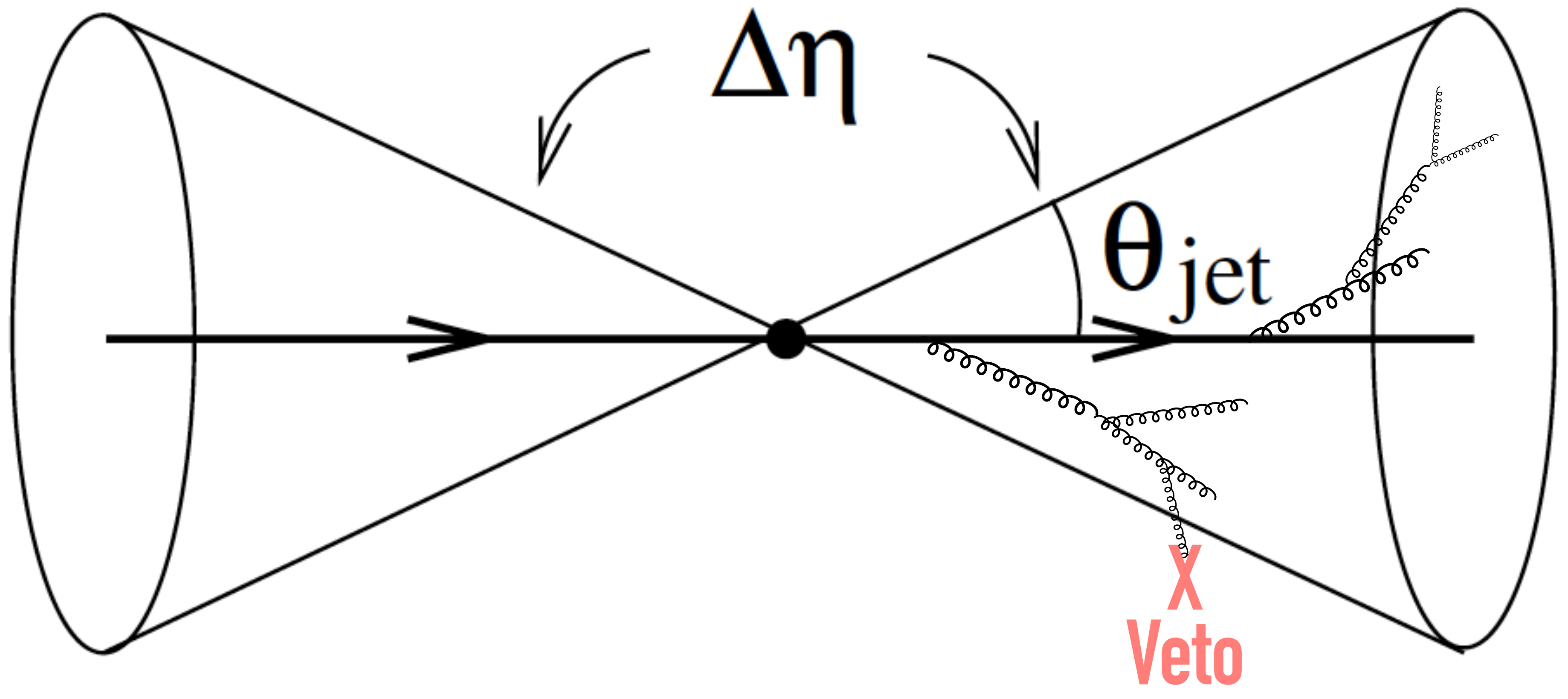

$$\Delta\eta := \ln\frac{1+c}{1-c}\,, \qquad c = \cos\theta_{jet}$$

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Need to calculate distribution of soft gluons on the sphere as a function of the veto scale

The toolkit: generating functional (GF) method

- Introduce generating functionals $Z_{12}[Q;\{u\}]$: distribution of soft radiation within the {12} dipole

see e.g. [Konishi, Ukawa, Veneziano ’79; Dokshitzer, Khoze, Mueller, Troyan ’91]

symmetry factor

generating functional

$$dP_n^{\{12\}} = \text{מ}(n)\left(\prod_{i=1}^{n}[dk_i]\frac{\delta}{\delta u(k_i)}\right) Z_{12}(Q,\{u\})\Big|_{\{u\}=0}$$

$$\frac{\delta}{\delta u(k_i)}u(k) \equiv \bar{\delta}(k-k_i)$$

“probing functions”

- E.g. global observables at LL: in e^+e^- case (e.g. thrust)

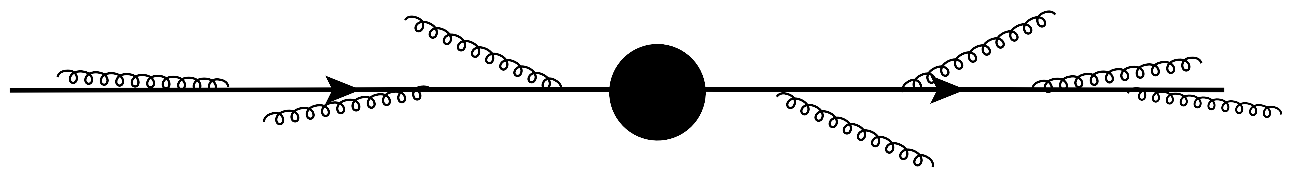
Angular ordering

$$Z_q(Q,\{u\}) = u(Q)\Delta_q(Q) + \int^{Q}[dk]\bar{\alpha}\,w^{(0)}(k)\frac{\Delta_q(Q)}{\Delta_q(k)}\underbrace{Z_q(Q-k,\{u\})}_{\simeq Q}\underbrace{Z_g(k,\{u\})}_{\simeq u(k)}\Theta_{\rm AO} \;\rightarrow\; Z_q(Q,\{u\}) = u(Q)\exp\left\{\int^{Q}[dk]\bar{\alpha}\,w^{(0)}(k)(u(k)-1)\right\}$$

Eikonal squared amp.

Soft-collinear primary radiation: linear evolution eqn.

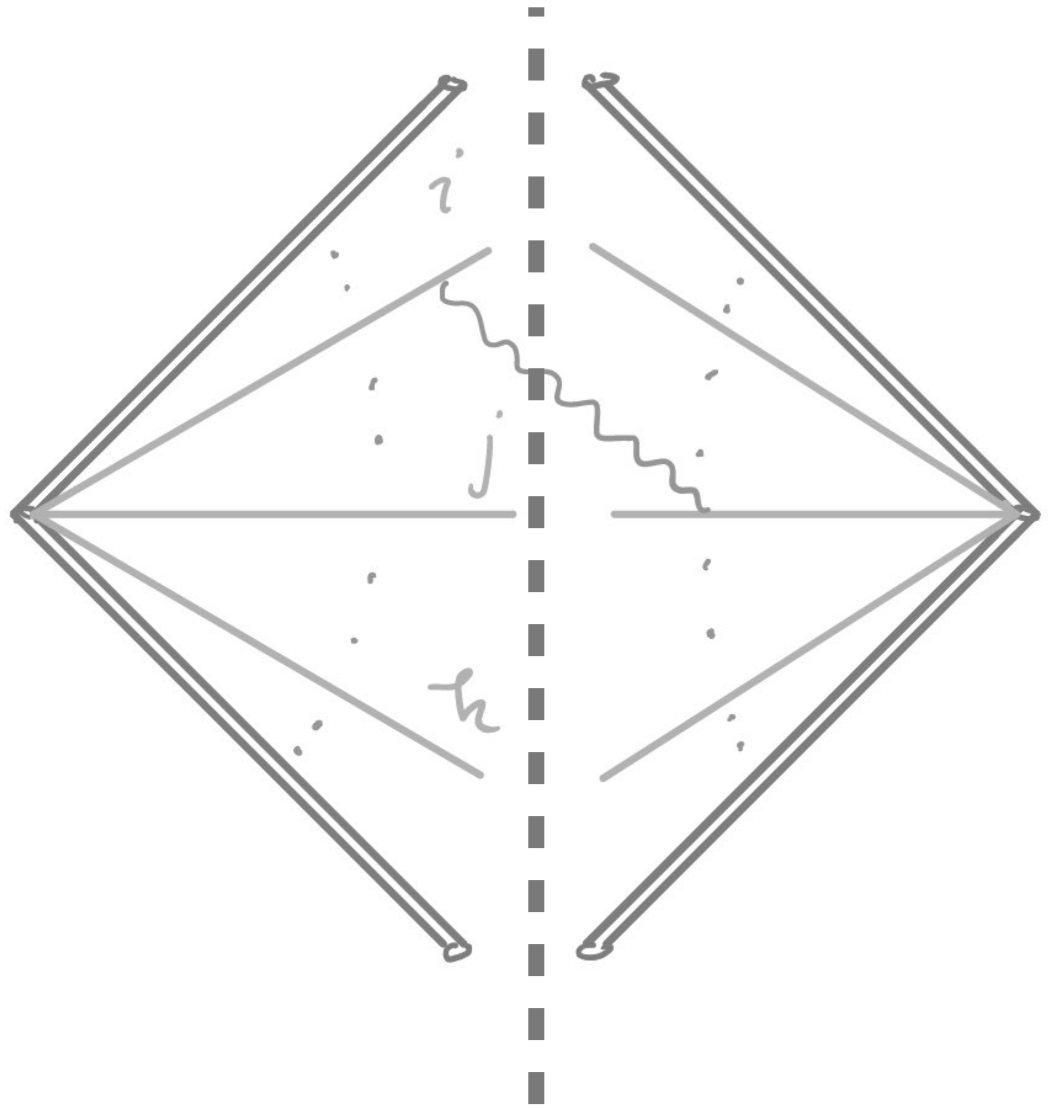
$$\sum_n dP_n = \sum_n \frac{1}{n!}\frac{d^n}{du^n}Z_q(Q,u)\Big|_{u=0} \sim \Delta_q(Q)\Big(1+\bar{\alpha}\,w^{(0)}(k_1)+\frac{\bar{\alpha}^2}{2!}w^{(0)}(k_1)w^{(0)}(k_2) + \frac{\bar{\alpha}^3}{3!}w^{(0)}(k_1)w^{(0)}(k_2)w^{(0)}(k_3)+\dots\Big)\,.$$



(non-linear) Evolution of the non-global GF

- Complexity growth of colour structure with any new emission: take large-N_c

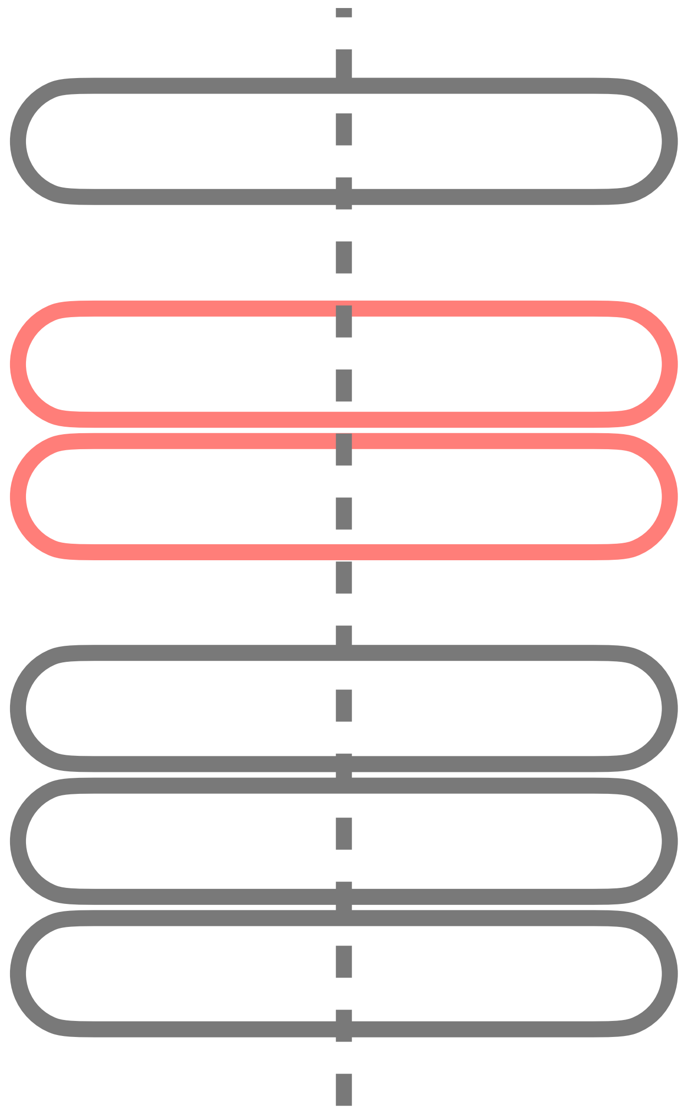
e.g. $O(\alpha_s)$ evolution (LL)



$\Rightarrow$

['t Hooft '73]

$\mathbf{T}_i \cdot \mathbf{T}_j \sim N_c\, \delta_{j,i\pm1}$



$$\mathcal{A}_{12}^2 = \bar{\alpha}^n(\mu)(2\pi)^{2n}(\mu^{2\epsilon})^n \sum_{\pi_n} \frac{(p_1 \cdot p_2)}{(p_1 \cdot k_{i_1})(k_{i_1} \cdot k_{i_2}) \ldots (k_{i_n} \cdot p_2)}$$

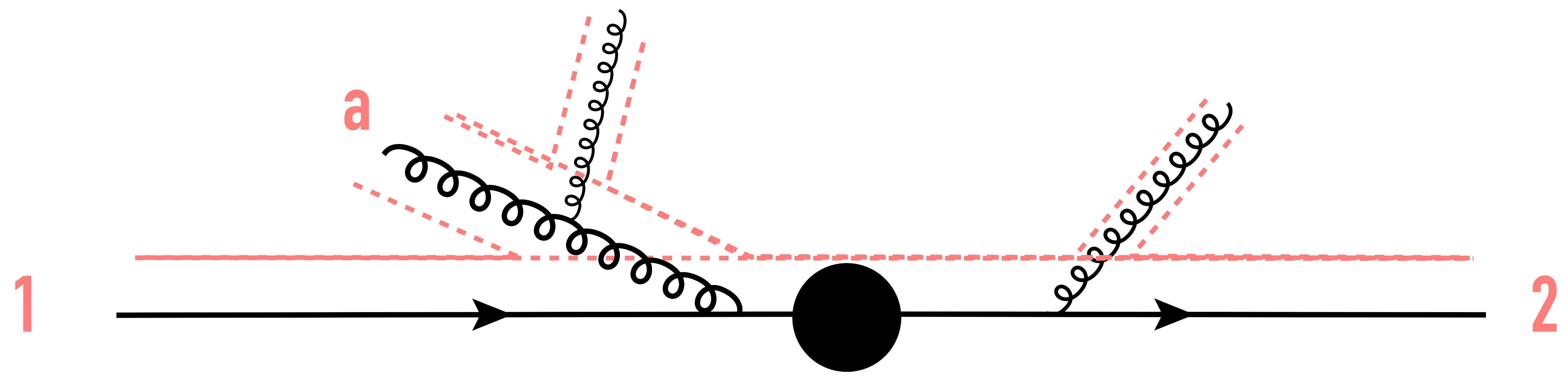
[Bassetto, Ciafaloni, Marchesini '83; Fiorani, Marchesini, Reina '88]

In the 't Hooft planar limit ($N_c \gg 1$, $\alpha_s N_c$ fixed) the evolution can be expressed as a closed equation at the level of the squared amplitude (i.e. colour dipoles) – treatable !

(non-linear) Evolution of the non-global GF

- Dependence on geometry cannot be handled analytically: use Markov chain Monte Carlo

e.g. O(α_s) evolution (LL)



Soft dipole radiation: non-linear evolution equation

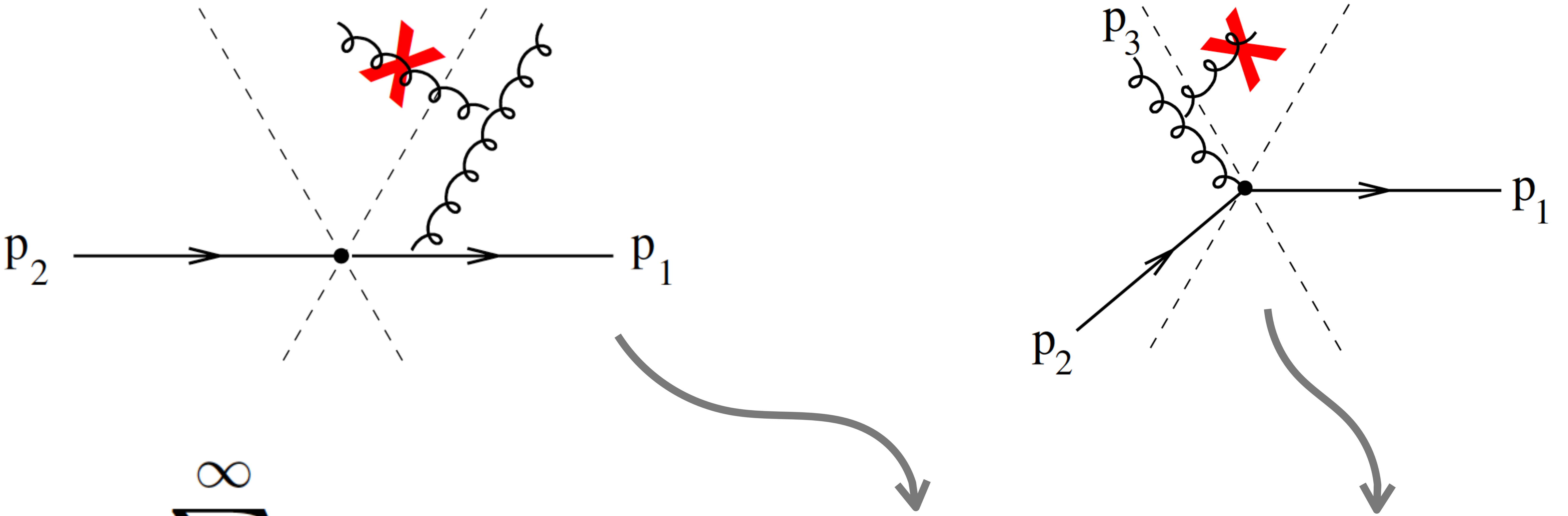
$$Z_{12}[Q;\{u\}] = \Delta_{12}(Q) + \int [dk_a]\bar{\alpha}(k_{ta})w_{12}^{(0)}(k_a)\,\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \times Z_{1a}[k_{ta};\{u\}]Z_{a2}[k_{ta};\{u\}]u(k_a)\Theta(Q-k_{ta})$$

Sudakov: no-emission probability (defined by $Z_{12}[Q;\{u=1\}] = 1$)

Symmetries of squared amplitude allow for an iterative reconstruction (strongly) ordered in dipole k_t

Factorisation of the cross section

- Cross section receives contributions from hard configurations with different multiplicity



$$\Sigma(v) := \sum_{n=2}^{\infty} \mathcal{H}_n \otimes S_n(v) = \mathcal{H}_2 \otimes S_2(v) + \mathcal{H}_3 \otimes S_3(v) + \cdots$$

Convolutions defined over solid angles in 4D (Hard and Soft factors are separately IRC finite)

$$\mathcal{H}_n \otimes S_n(v) = \int \left(\prod_{i=1}^{n} d^2\Omega_i \right) \mathcal{H}_{1\ldots n} \times S_{1\ldots n}(v)$$

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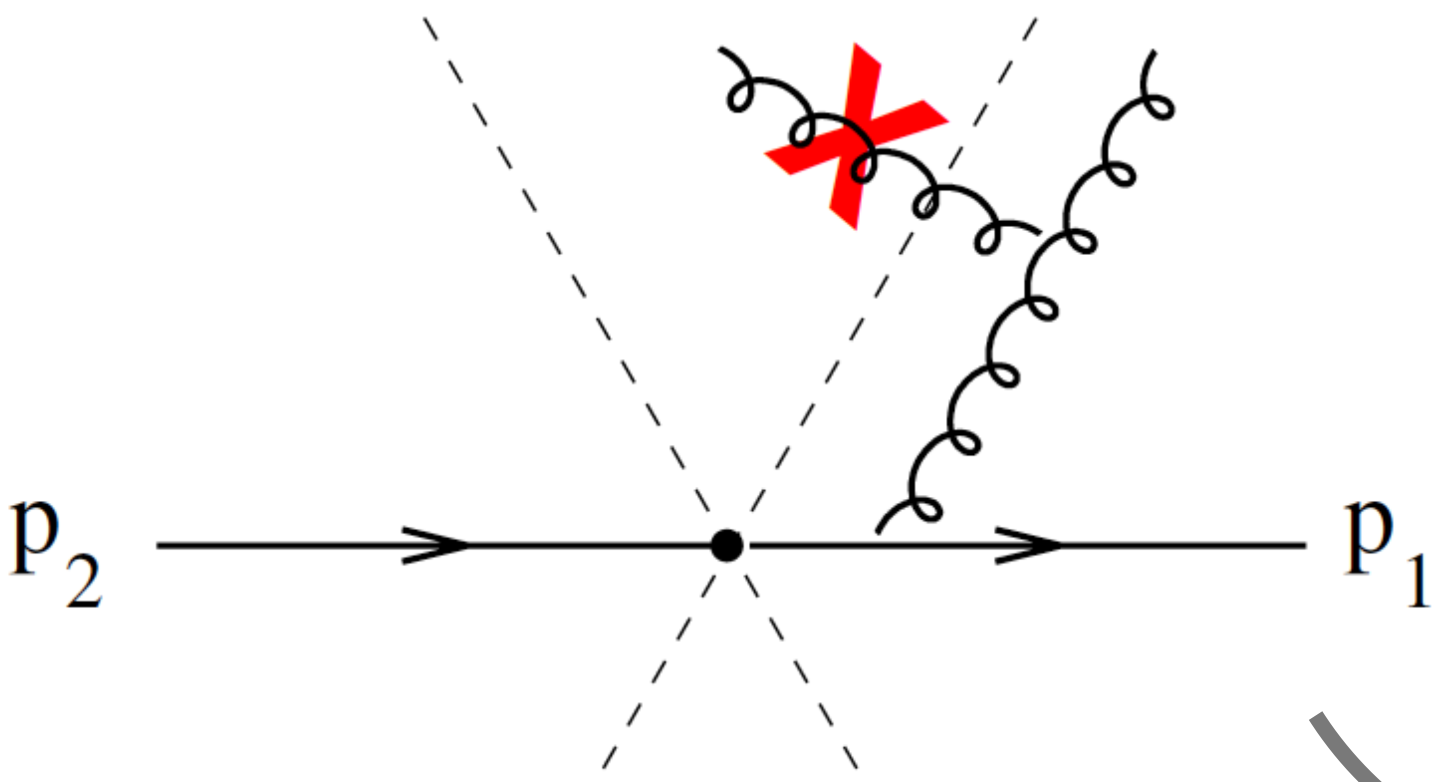

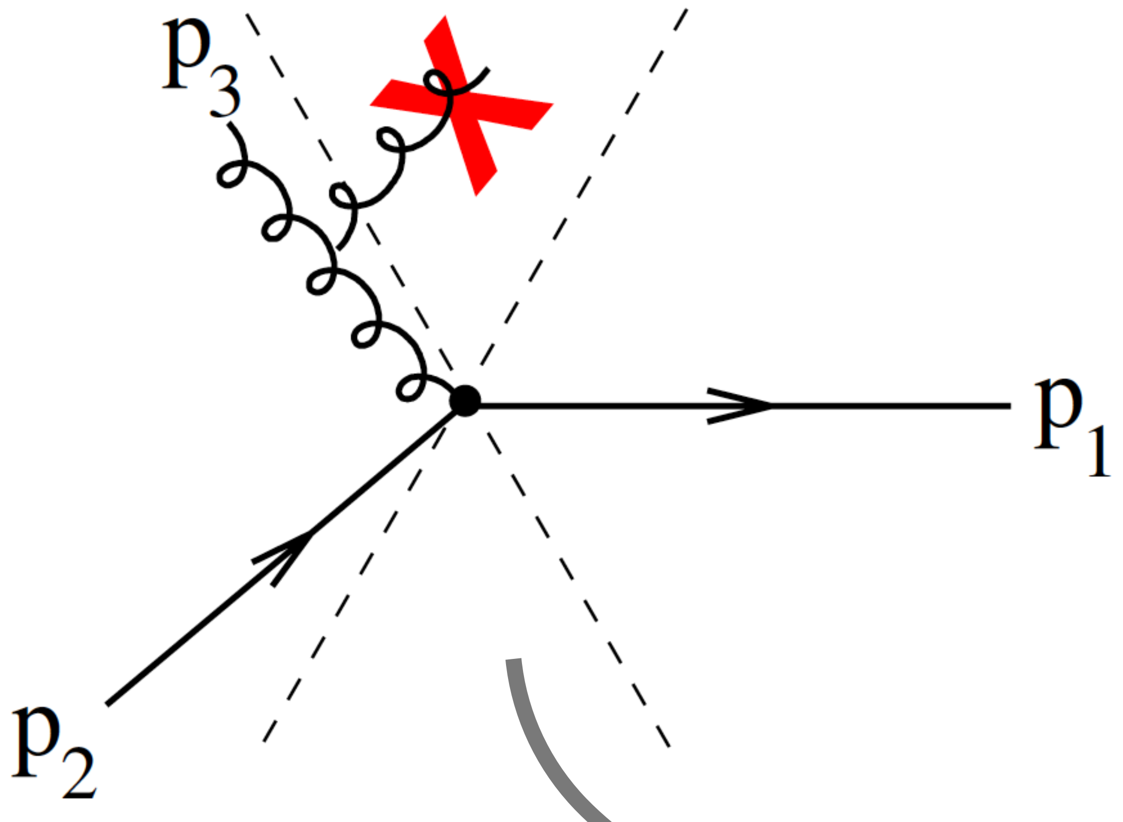

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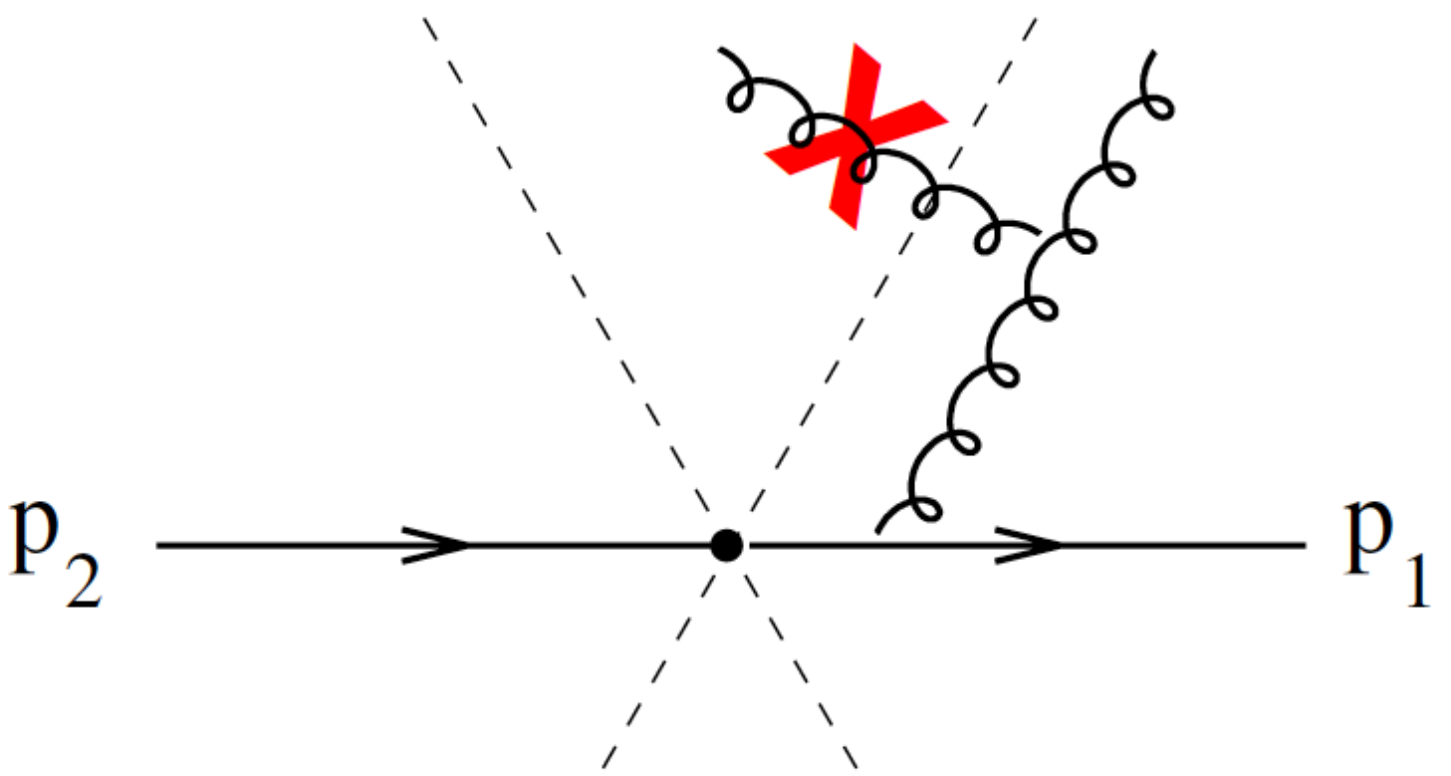
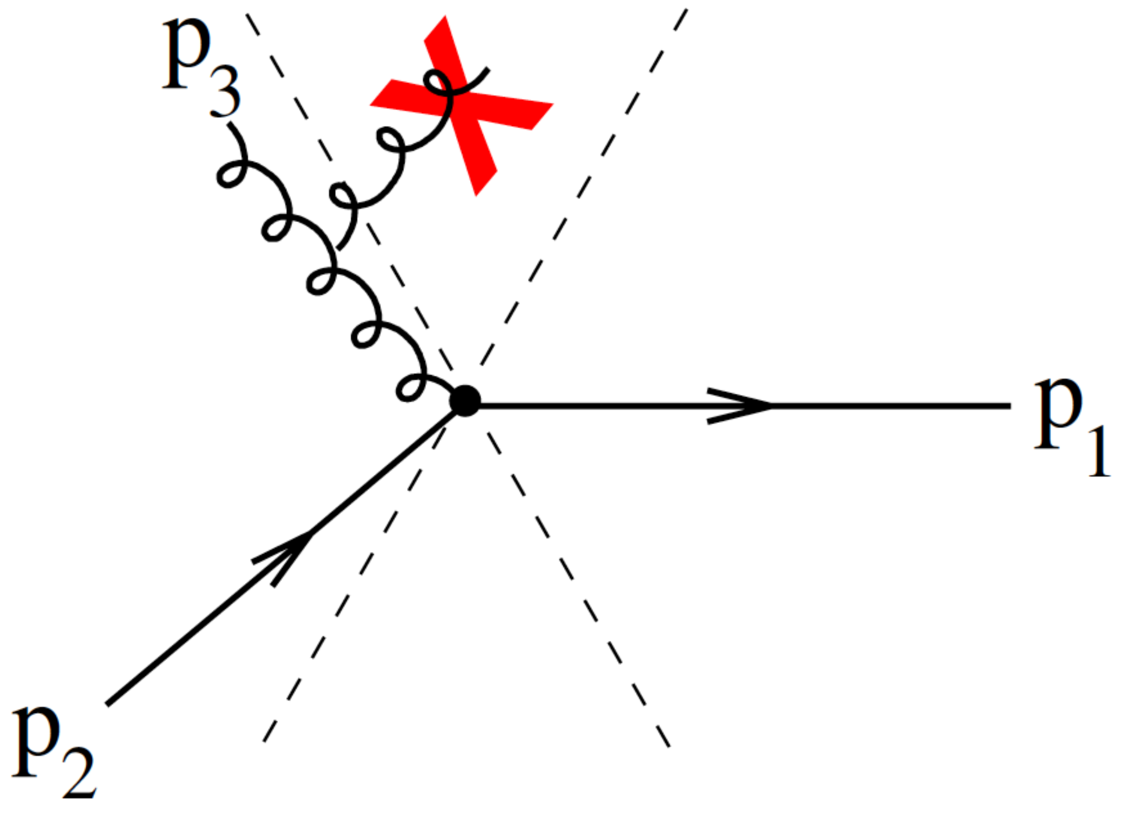
	$\mathcal{H}_2$ ↓	$S_2(v)$ ↓	$\mathcal{H}_3$ ↓	$S_3(v)$ ↓
LL	LO	$\mathcal{O}(\alpha_s)$ evolution	-	-
NLL	NLO	$\mathcal{O}(\alpha_s^2)$ evolution	LO	$\mathcal{O}(\alpha_s)$ evolution
	⋮			

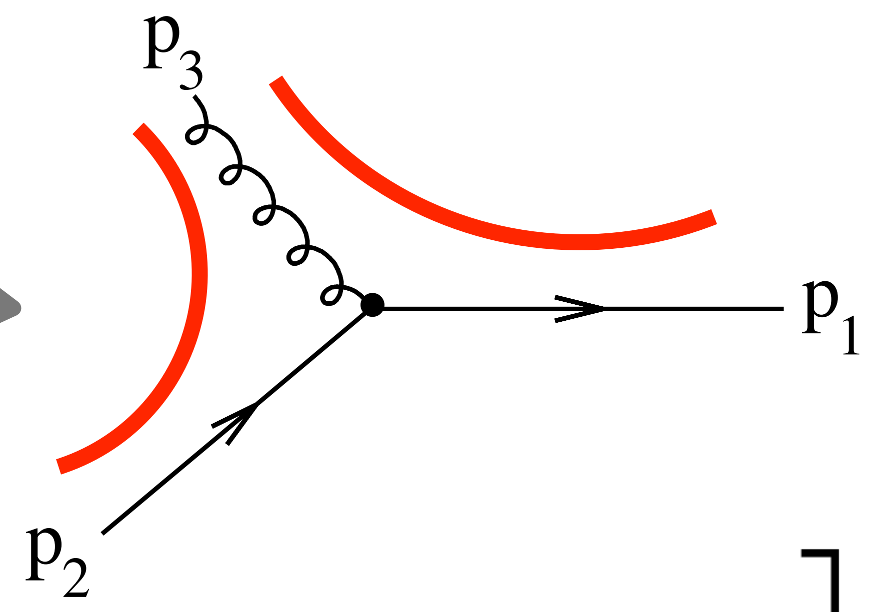
Factorisation of the cross section

- Factorisation theorem simplifies in the planar limit (in GFs language)





$N_c \gg 1$



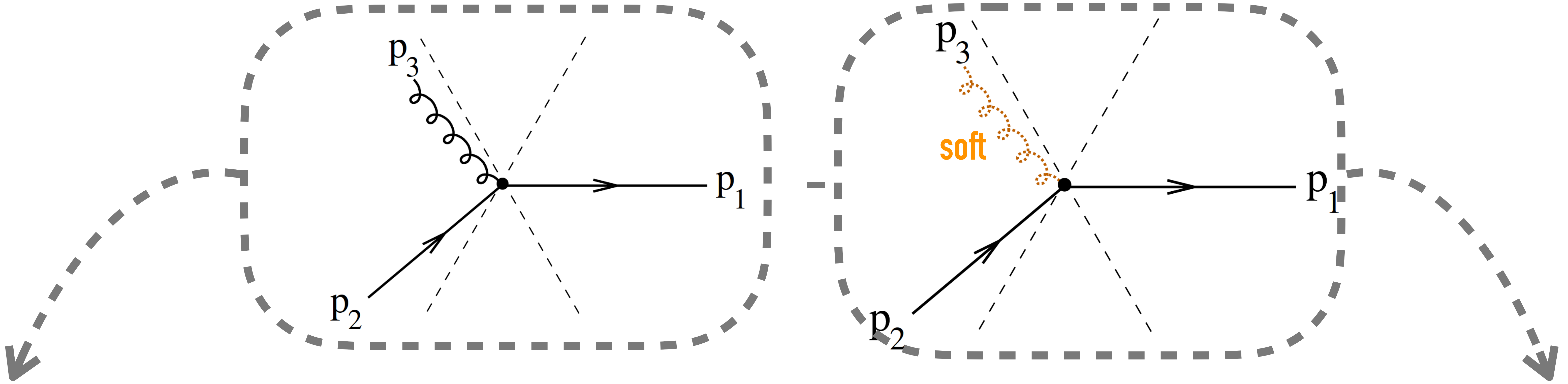
$$\Sigma(v) = \mathcal{H}_2 \otimes \left[\sum_{i=0}^{\infty} \int dP_i^{\{12\}} \Theta(v - V(\{k_i\})) \right]$$
$$+ \mathcal{H}_3 \otimes \left[\left(\sum_{i=0}^{\infty} \int dP_i^{\{13\}} \right) \left(\sum_{j=0}^{\infty} \int dP_j^{\{23\}} \right) \Theta(v - V(\{k_i\}, \{k_j\})) \right] + \mathcal{O}(\text{NNLL})$$

Hard factors at NLL

Recall:
$$\mathcal{H}_n \otimes S_n(v) = \int \left(\prod_{i=1}^{n} d^2\Omega_i \right) \mathcal{H}_{1\ldots n} \times S_{1\ldots n}(v)$$

- Computed by matching the soft theory to full QCD
- Cancellation of collinear divergences between H_2 and H_3 (only combination is scheme indep.!)

e.g. H_3



Subtract counter-term (2-jet kinematics) with full ME, requiring all partons to be outside the slice. Thrust axis along the hardest parton

Subtract soft counter-term, requiring the soft gluon to be outside the slice. Thrust axis along q (qbar) direction

Hard factors at NLL

Recall:
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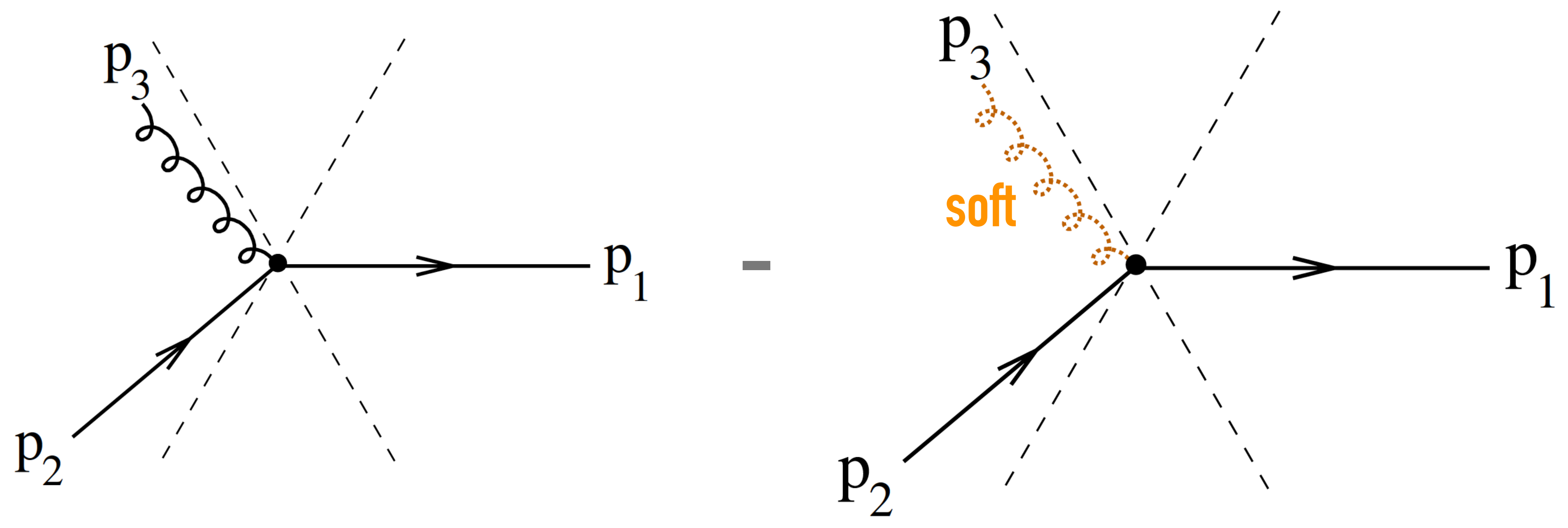

- Event and counter-event are then dressed by the soft factors S_3 and S_2, respectively

$$\mathcal{H}_3 \otimes S_3(v) = \Sigma^{(3),\mathrm{sub}}(v) - \Sigma^{(3),\mathrm{sub}}_{\mathrm{soft}}(v)$$

All-order formulation of Projection-to-Born subtraction

Adapted from [Cacciari, Dreyer, Karlberg, Salam, Zanderighi '15]

Hard factors at NLL

Recall:

$$\Delta\eta := \ln \frac{1+c}{1-c}\,, \qquad c = \cos\theta_{\text{jet}}$$

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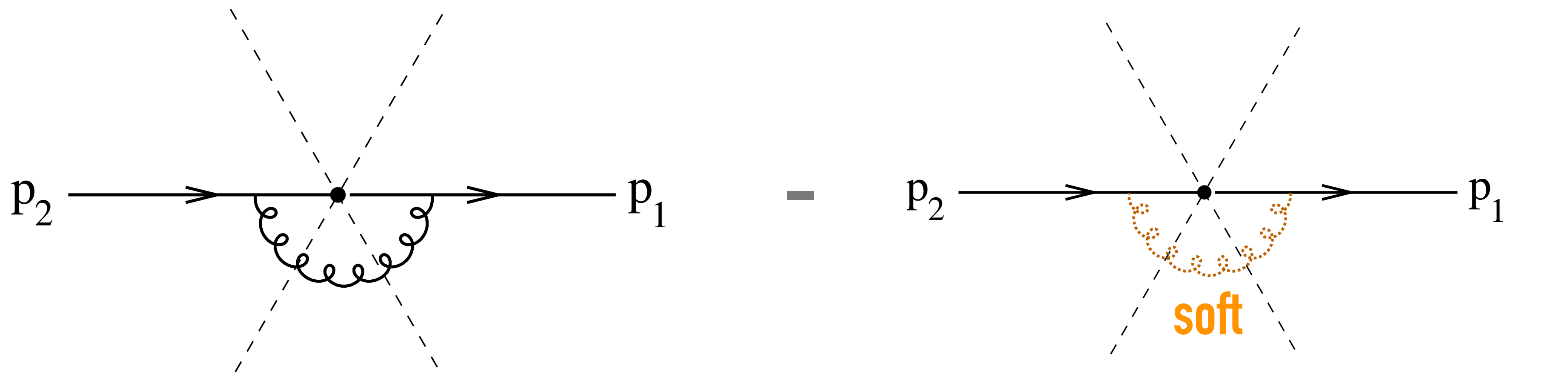
- Cancellation of collinear divergences between H_2 and H_3 (only combination is scheme indep.!)

e.g. H_2:



integrated counterterms

$$\Rightarrow \quad \mathcal{H}_2 = \delta^{(2)}(\Omega_1 - \Omega_q)\delta^{(2)}(\Omega_2 - \Omega_{\bar{q}})\left(1 + \frac{\alpha_s}{2\pi}\mathcal{H}_2^{(1)}\right)$$

$$\begin{aligned}\mathcal{H}_2^{(1)} = \frac{C_F}{2\left(1-c^2\right)^2}\Bigg(& 4\left(1-c^2\right)^2\left(\mathrm{Li}_2\left(\frac{1+c}{2}\right) - \mathrm{Li}_2\left(\frac{1-c}{2}\right)\right) \\ &- 2\left(1-c^2\right)^2\log^2(1+c) + 16c\left(3+c^2\right)\ln(2) - (1-c^2)(c(16+3c)-3) \\ &+ 2\ln(1-c)\left(-2\left(1+c^4\right)\log(2) - 4c\left(3+c^2\right) + \left(1-c^2\right)^2\ln(1-c)\right) \\ &+ \left(4\left(1+c^4\right)\ln(2) - 8c\left(3+c^2\right)\right)\ln(1+c) - 4\left(-3c^4 + 2c^2(9+2\ln(2)) + 1\right)\tanh^{-1}(c)\Bigg)\end{aligned}$$

Second-order (planar) corrections to evolution kernel

- NLL evolution kernel describes $O(\alpha_s^2)$ soft gluon exchanges within each colour dipole

$$Z_{12}[Q;\{u\}] = \Delta_{12}(Q) + \int [dk_a]\bar{\alpha}(k_{ta}) w_{12}^{(0)}(k_a)\,\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \times Z_{1a}[k_{ta};\{u\}] Z_{a2}[k_{ta};\{u\}] u(k_a)\Theta(Q-k_{ta})$$

$$\Downarrow$$

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

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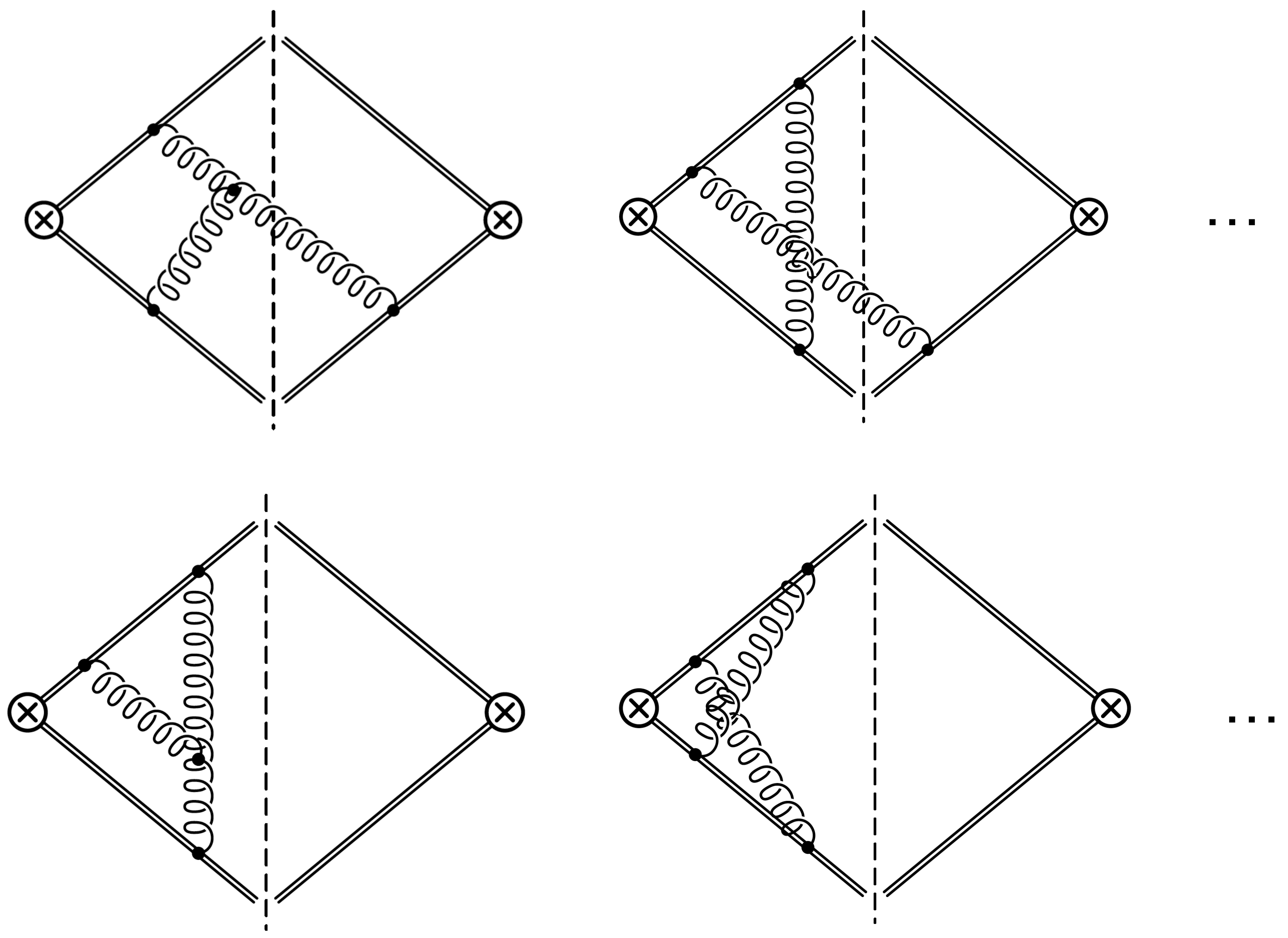
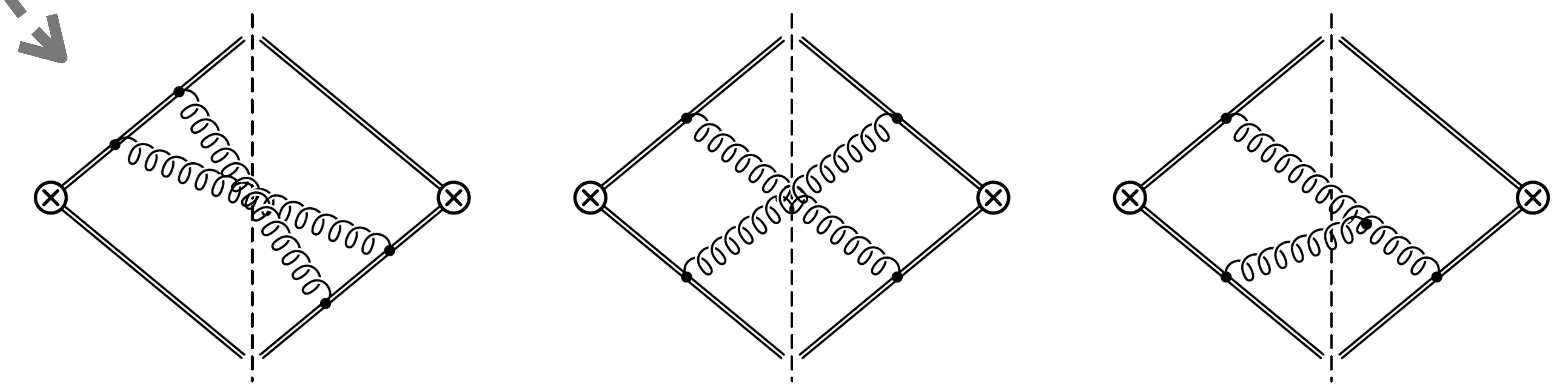
subtraction of iteration of LL kernel
(no double counting)



two <u>unordered</u> real gluons



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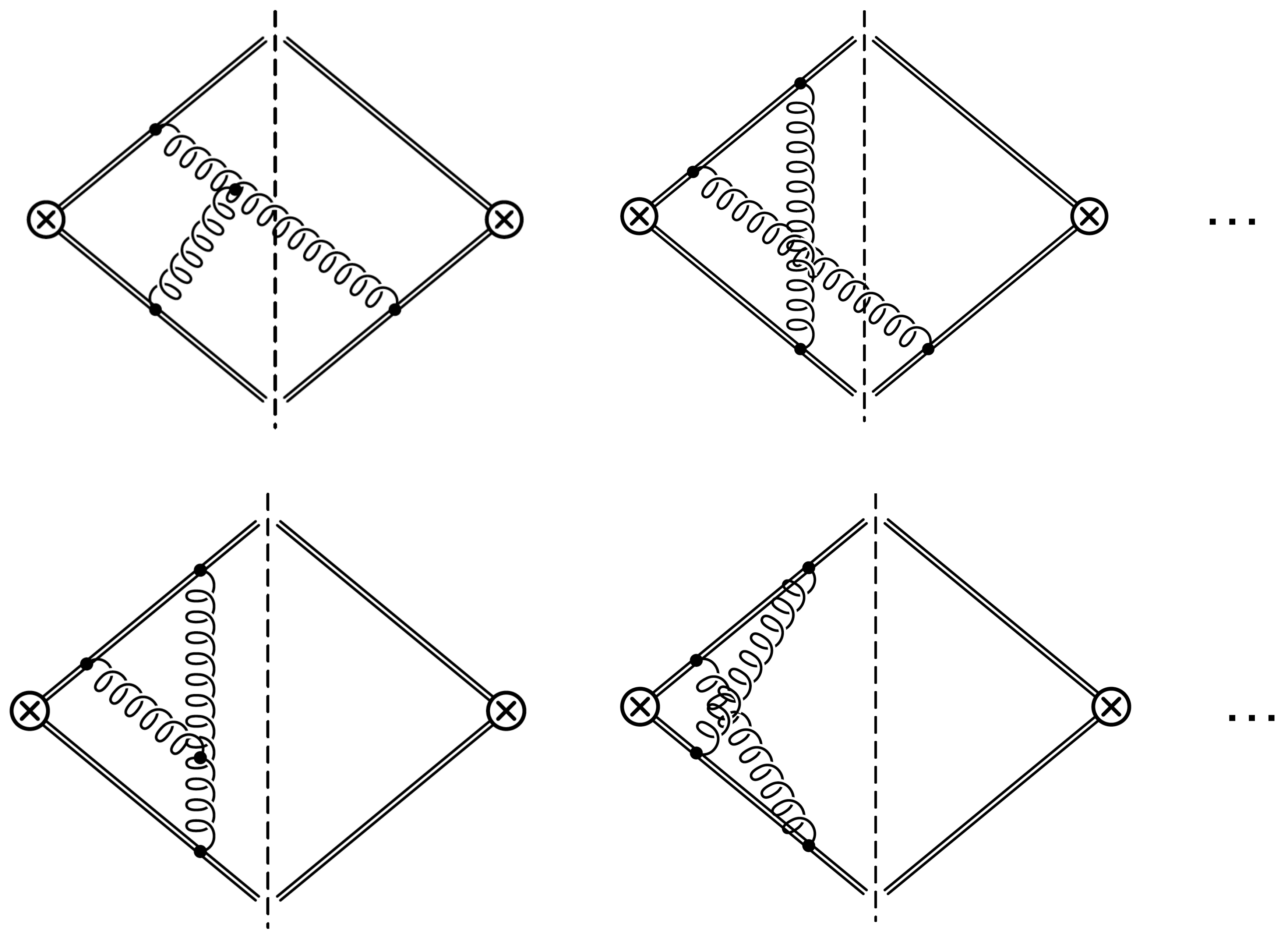
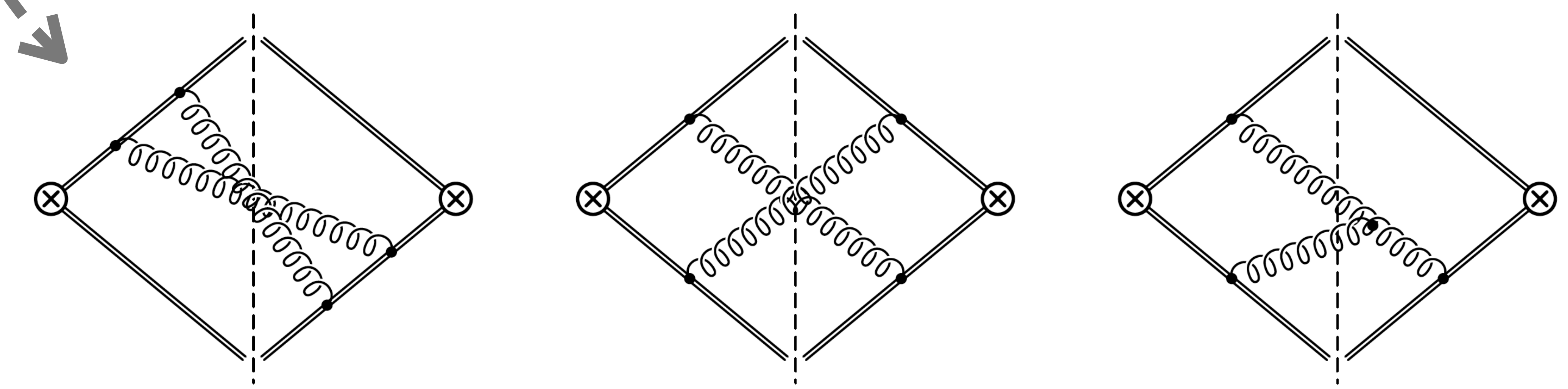


\+ integrated counter-terms

two unordered real gluons



\- local counter-term

Introduce IRC counter-term to make the NLL kernel calculable in 4 dimensions

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

two-loop cusp anomalous dimension

$$\mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] = \Delta_{12}(Q) + \int [dk_a]\bar\alpha(k_{ta}) w_{12}^{(0)}(k_a)\left(1+\bar\alpha(k_{ta})\,\bar K^{(1)}\right)\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \times Z_{1a}[k_{ta};\{u\}]Z_{a2}[k_{ta};\{u\}]u(k_a)\Theta(Q-k_{ta})$$

Same structure as LL kernel

(1→2 dipole branching)

Easy to iterate in a MCMC

Squared amplitudes from [Catani, Grazzini '00]
also [Angeles Martinez, Forshaw, Seymour '16]

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z[Q;u],u] + \mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] - \mathbb{K}_{\rm int}^{\rm DC}[Z[Q;u],u]$$

$$\mathbb{K}_{\rm int}^{\rm RR}[Z[Q;u],u] = \int [dk_a] \int [dk_b]\, \bar{\alpha}^2(k_{t(ab)})\Theta(Q-k_{t(ab)})\Theta(k_{ta}-k'_{tb})\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{t(ab)})}$$
$$\times \Big[\bar{w}_{12}^{(gg)}(k_b,k_a)Z_{1b}[k_{t(ab)};\{u\}]Z_{ba}[k_{t(ab)};\{u\}]Z_{a2}[k_{t(ab)};\{u\}]u(k_a)u(k_b)$$
$$+\bar{w}_{12}^{(gg)}(k_a,k_b)Z_{1a}[k_{t(ab)};\{u\}]Z_{ab}[k_{t(ab)};\{u\}]Z_{b2}[k_{t(ab)};\{u\}]u(k_a)u(k_b)$$
$$-\left(\bar{w}_{12}^{(gg)}(k_b,k_a)+\bar{w}_{12}^{(gg)}(k_a,k_b)\right)Z_{1(ab)}[k_{t(ab)};\{u\}]Z_{(ab)2}[k_{t(ab)};\{u\}]u(k_{(ab)})\Big]$$

New structure of real radiation

(1 → 3 dipole branching)

Hard to iterate in a MCMC

collinear counter-term defined on a projected pseudo-parent momentum

Squared amplitudes from [Campbell, Glover '97]
also [Gehrmann-De Ridder, Gehrmann, Glover '05]

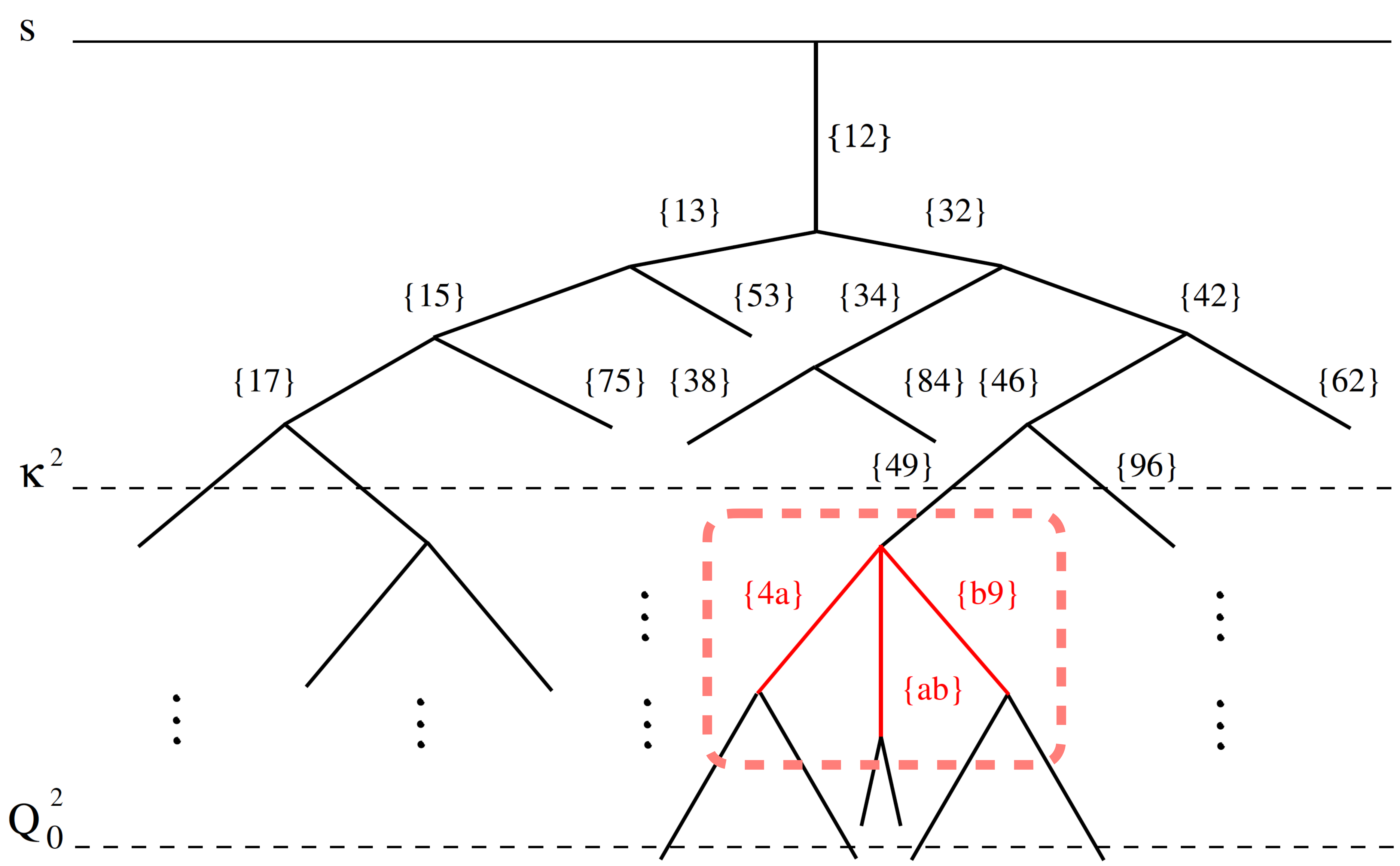
Perturbative solution of NLL evolution equation

- All-order solution can be formulated in a perturbative form, i.e.

$$Z_{12}[Q;\{u\}] = Z_{12}^{(0)}[Q;\{u\}] + Z_{12}^{(1)}[Q;\{u\}] \quad \text{with} \quad Z_{12}^{(0)}[Q;\{u\}] = \mathbb{K}_{\rm int}^{\rm RV+VV}[Z^{(0)}[Q;u],u]$$

- Linearise evolution equation in $Z^{(1)}$ by neglecting $(Z^{(1)})^2$ corrections (NNLL and higher)

[backup]

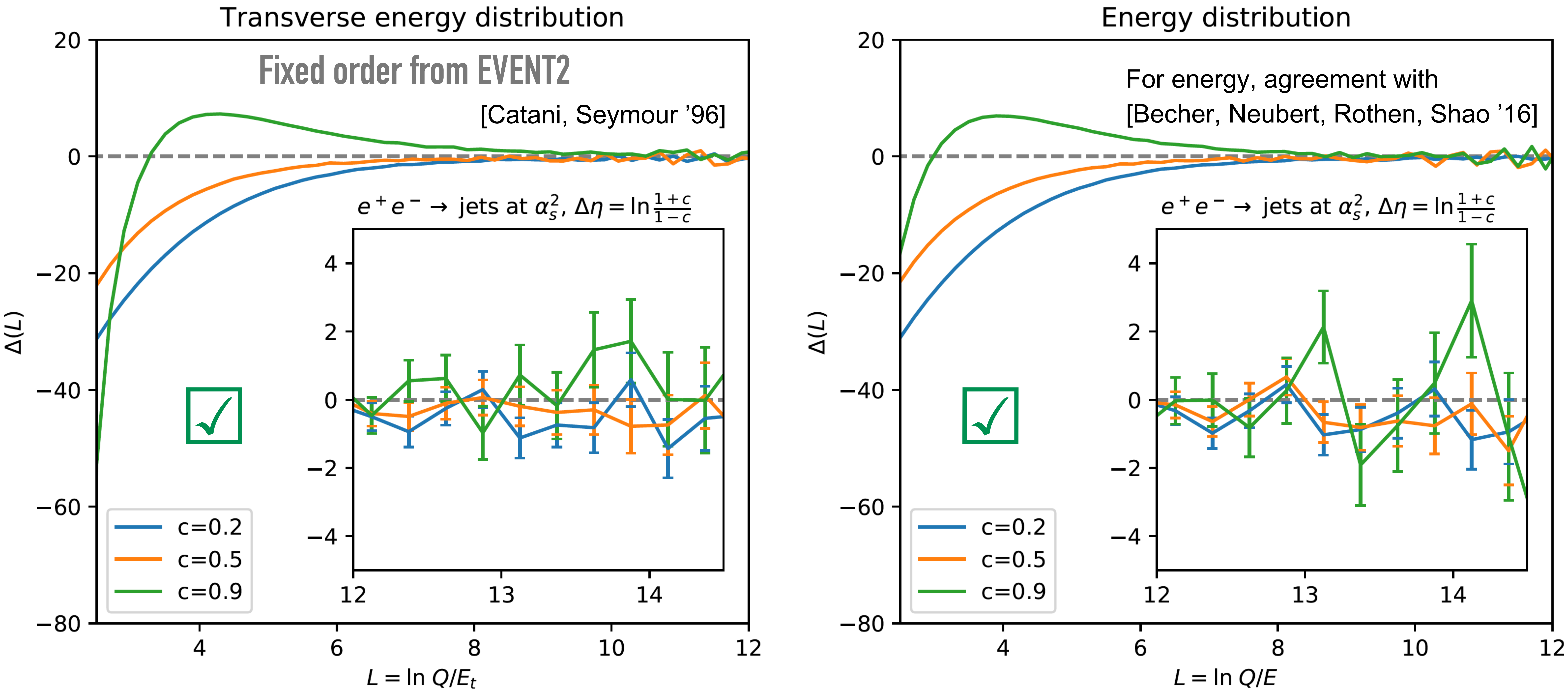


All-order iteration of $Z^{(0)}$ and a single insertion of $Z^{(1)}$ at <u>any scale</u> in the evolution graph (truncated shower). Structure emerges from the ev. eqn.

Fixed order expansion & checks

- $O(\alpha_S^2)$ expansion expected to reproduce the logarithmic structure of QCD

$$\Delta(L) := \frac{1}{\sigma_0}\left(\frac{d\Sigma^{\text{NLO}}}{dL} - \frac{d\Sigma^{\text{EXP.}}}{dL}\right) \qquad \text{expect} \qquad \lim_{L\to\infty} \Delta(L) = 0$$



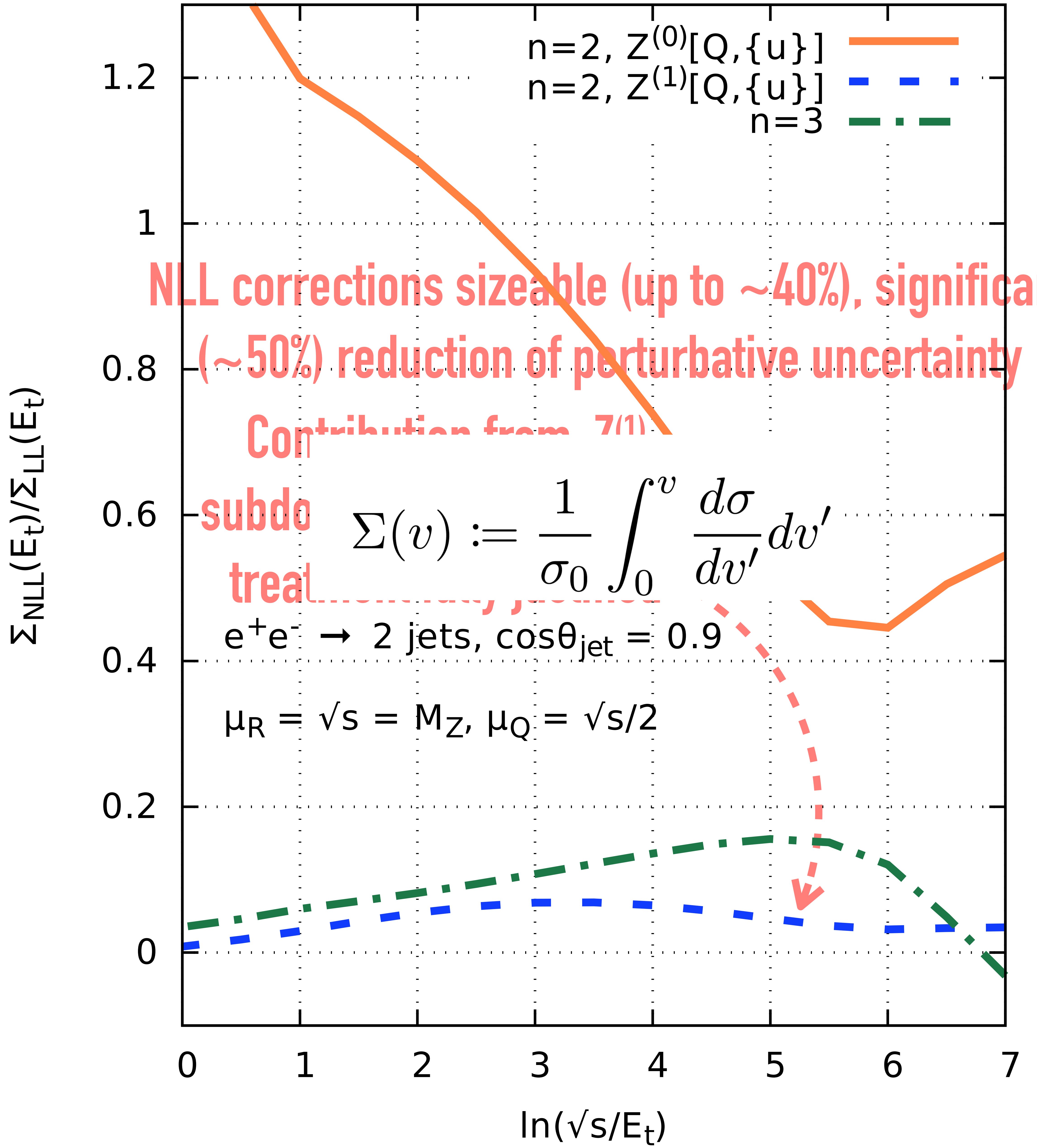
OK for different jet-cone sizes

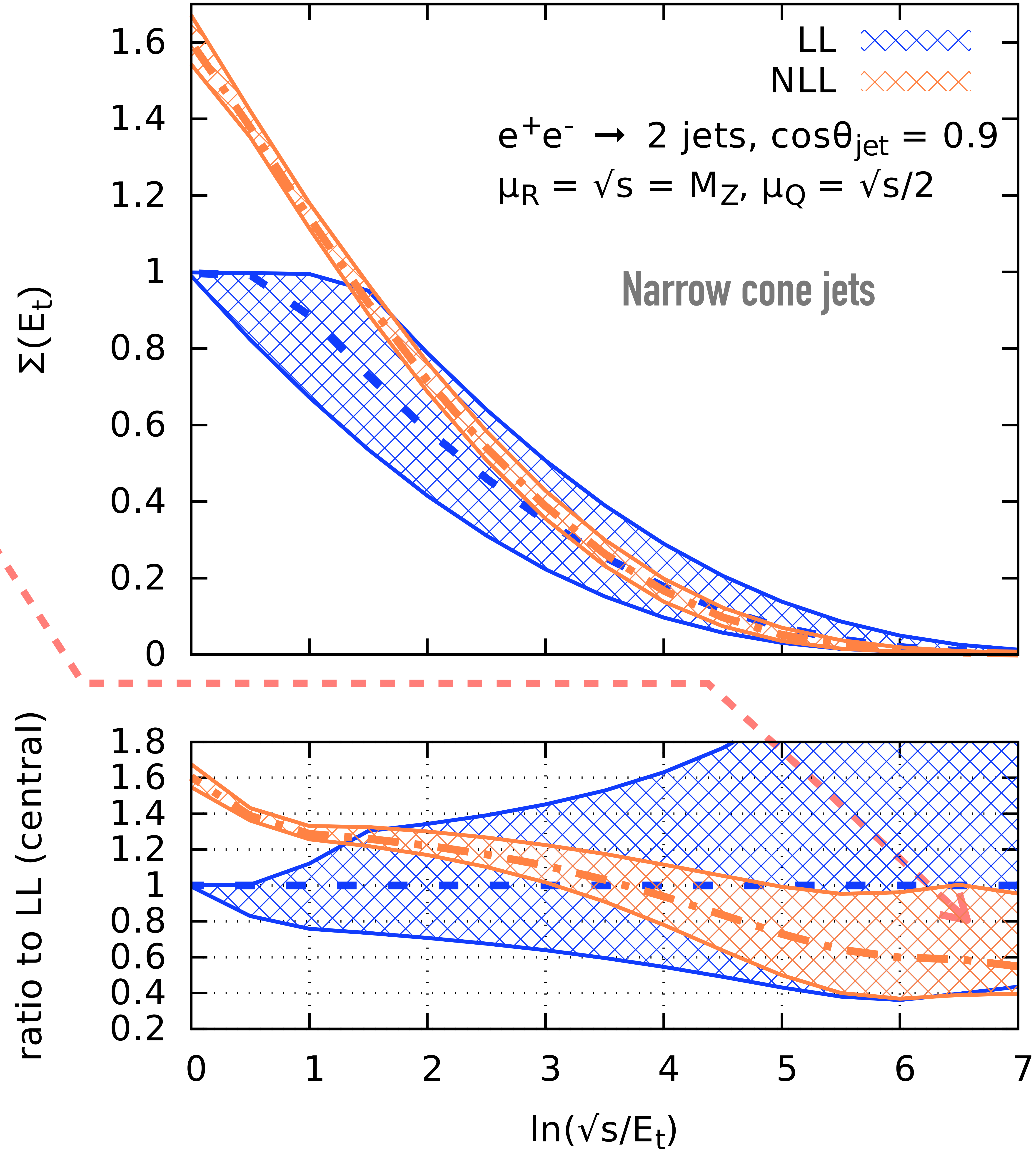
All-order results at NLL: narrow jets

NLL corrections sizeable (up to ~40%), significant (~50%) reduction of perturbative uncertainty

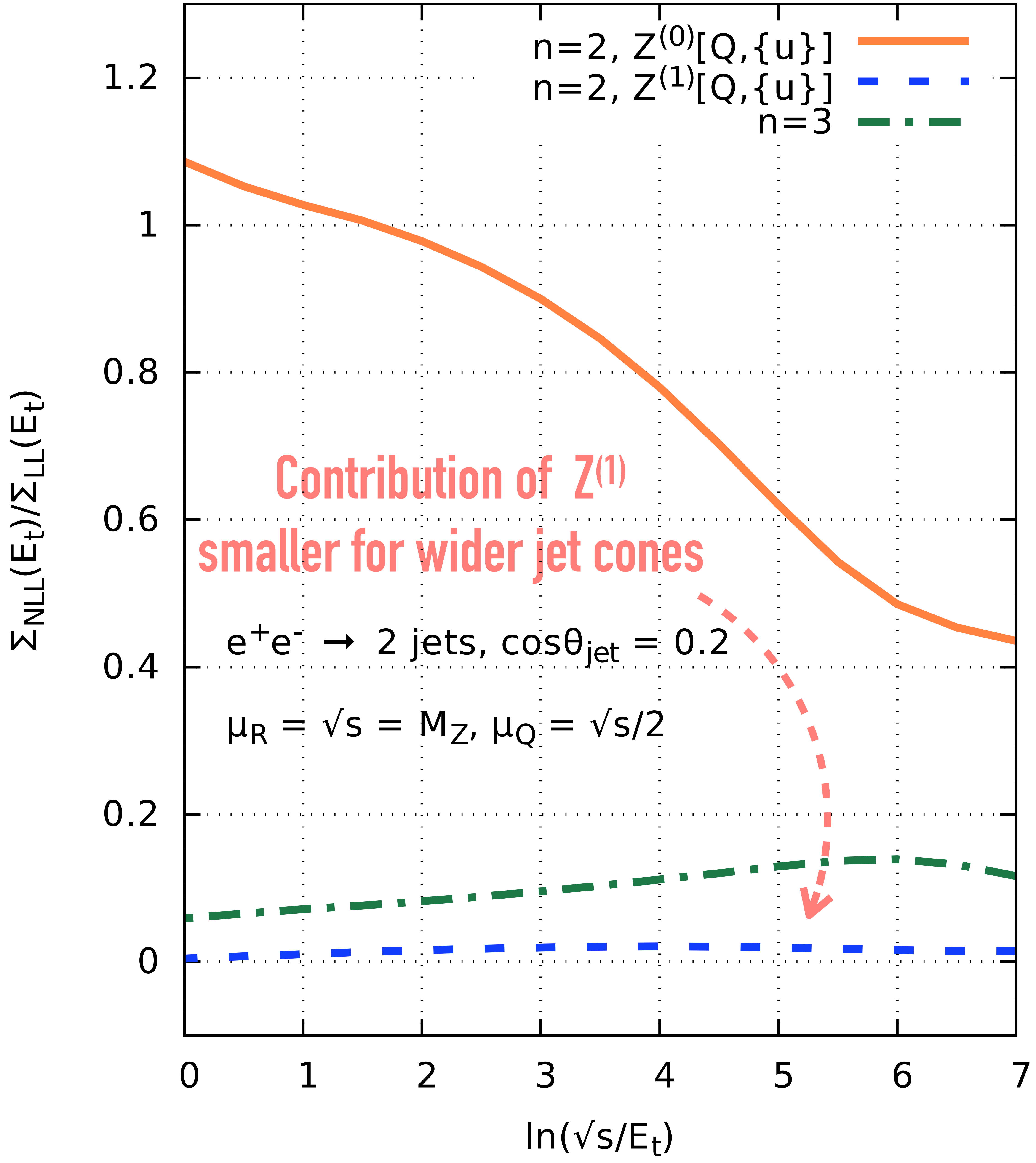
Contribution from $Z^{(1)}$ … subdo… treat…

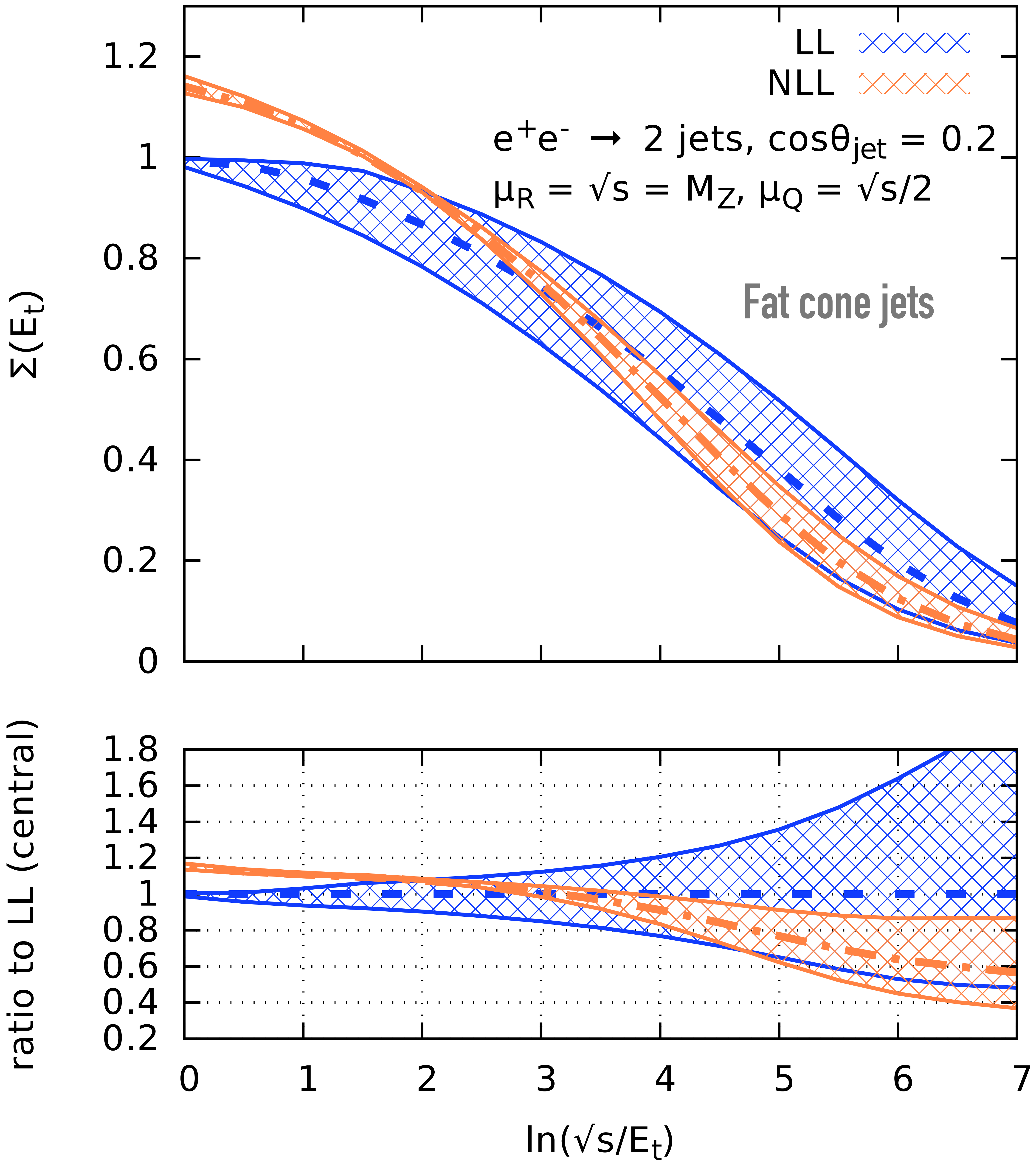
$$\Sigma(v) := \frac{1}{\sigma_0}\int_0^v \frac{d\sigma}{dv'}dv'$$





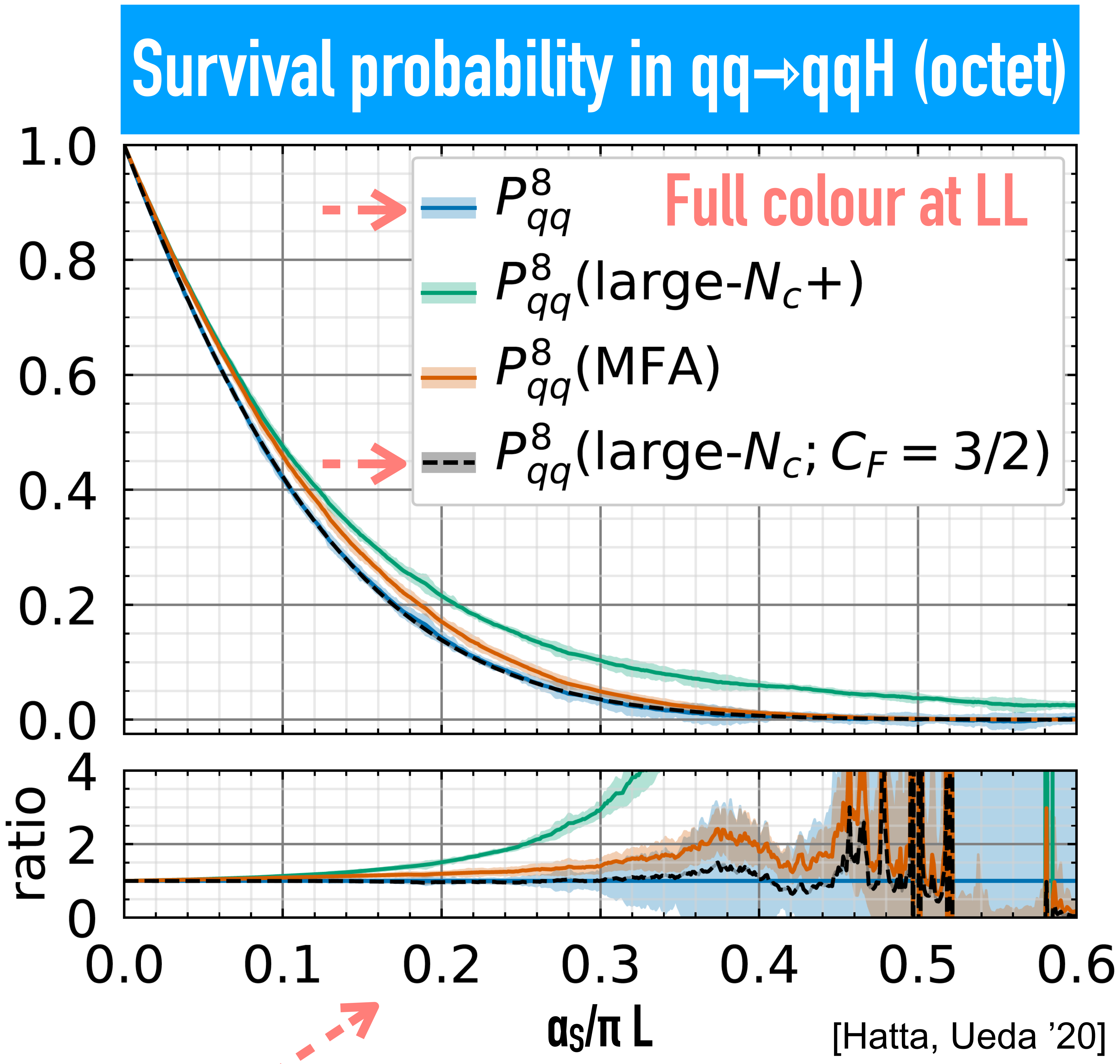
All-order results at NLL: fat jets

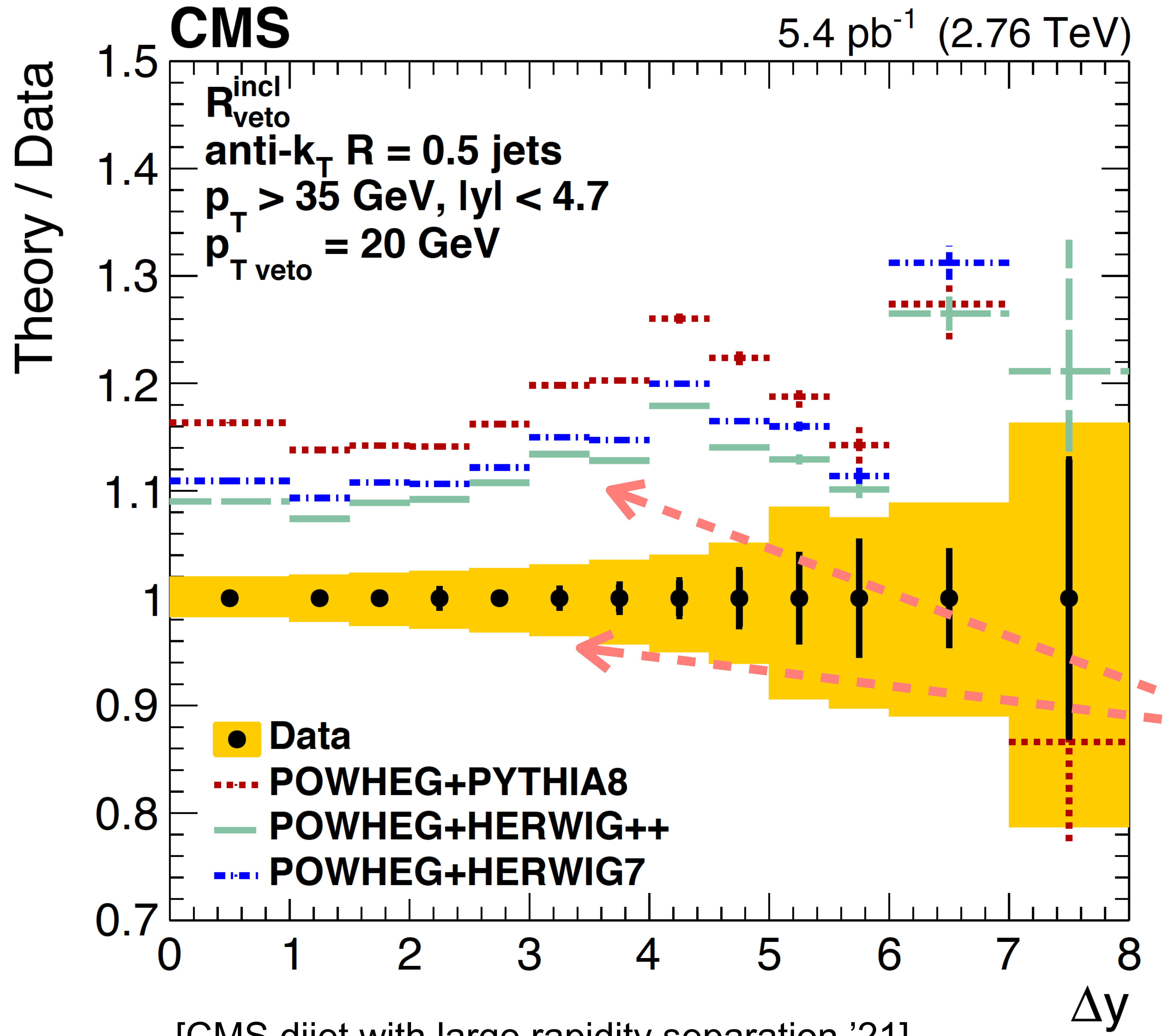




Outlook: applications to LHC phenomenology

- Leading-N_c method can be applied to hadronic collisions (subleading Nc sensitive to Glauber modes / super-leading logarithms). Several applications at the LHC, e.g.





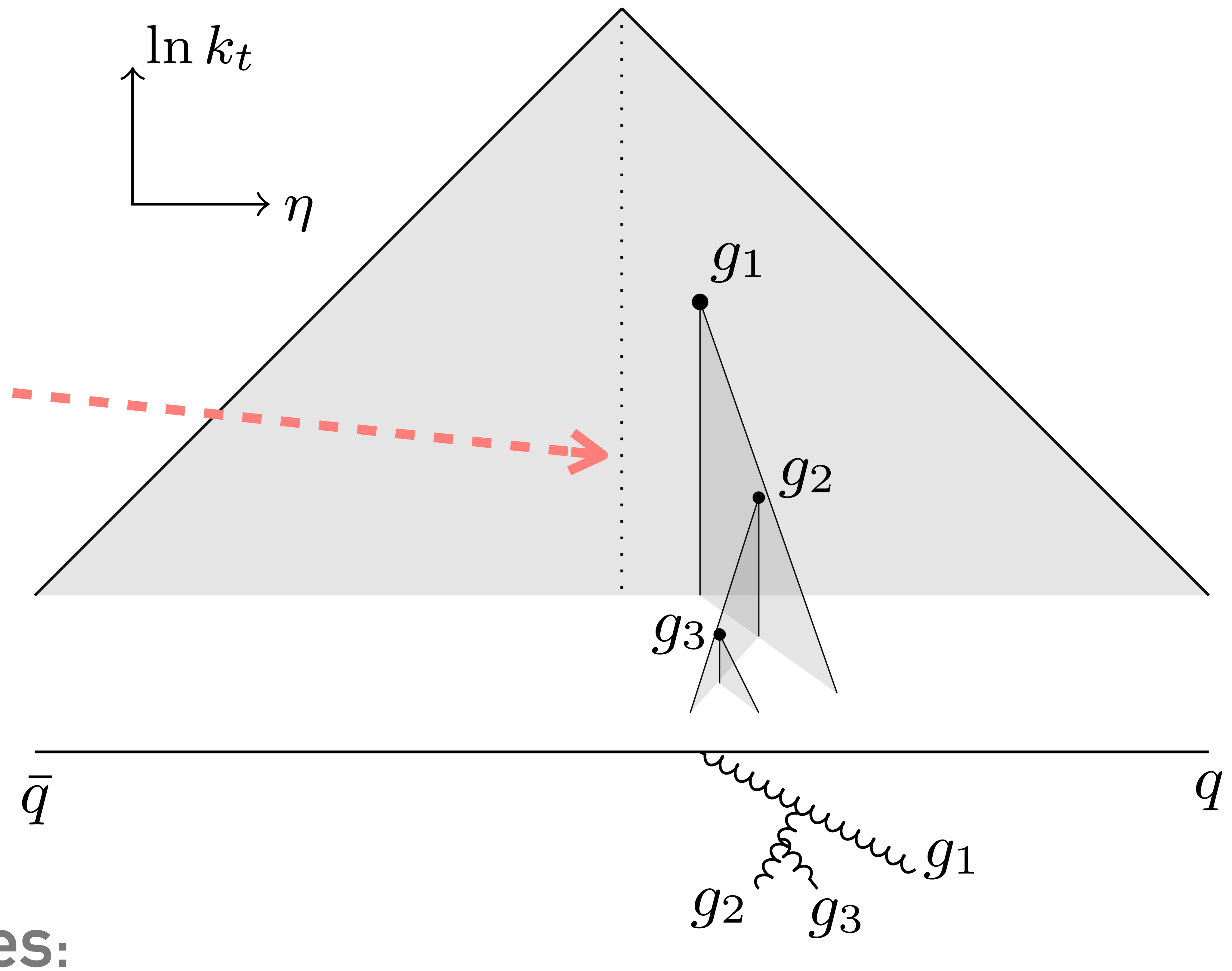
[CMS dijet with large rapidity separation '21]

- Higgs VBF vs. ggF discrimination via a 3rd jet veto
- Study of Mueller-Navelet jets

see also [CMS+TOTEM 2102.06945]

- Also jet mass, substructure observables, isolation, …

Outlook: aspects of a NNLL (soft) shower



- Technology relevant for description of soft radiation at wide angles in dipole showers (current algorithms reach at best LL for NGLs)
- NLL evolution equations provide guidance towards a NNLL algorithm (NGLs start at NLL in general)
- Problem is well defined, but several technical challenges:
  - Kinematics: iteration of $Z^{(1)}$ correction, maps for collinear counter-terms & recoil scheme
  - Ordering and phase space coverage: radiation of unordered pair within $Z^{(1)}$
  - Matching to NLO hard scattering (corrections to H_2 & H_3) while preserving NLL accuracy

Related work also in
[Plaetzer, Ruffa ’20; Dulat, Hoeche, Prestel ’18]

Conclusions

- New formalism for calculation of non-global corrections at NLL in the planar limit:
  - Soft evolution for GFs solvable in terms of colour dipoles with Monte Carlo methods
  - First NLL resummation for final-state radiation (veto in interjet rapidity gap). NLL corrections are substantial (up to 40%), with a considerable reduction of TH errors (~50%)
- In the planar approximation this technology can be applied to pp collision too, avoiding complications that would arise at sub-leading N_c (e.g. SLL / Glauber modes, fact. breaking)

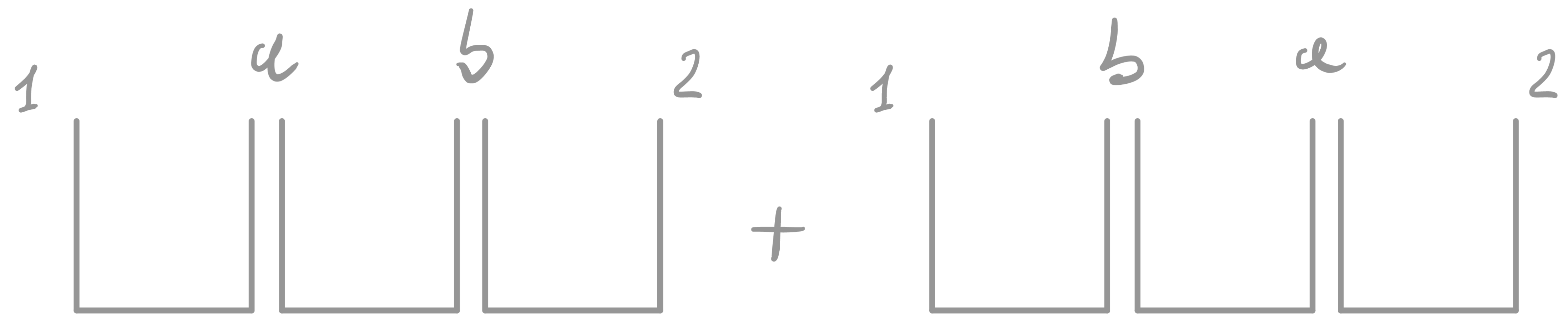
see e.g. [Forshaw, Kyrieleis, Seymour '06 - '08; Delgado, Forshaw, Marzani, Seymour '11; Becher, Neubert, Shao '21]

  - Process dependence encoded in the hard factors and calculation can be made algorithmic
- MCMC algorithm closely related to a parton shower: important insight on how to resum higher-orders NGLs in future generation of algorithms

Extra material

Evolution of the soft factors in a leading-N_c Monte Carlo

- Symmetries of the LL squared amplitude, e.g. 2 emissions



$$|A|^2 \sim \underbrace{\frac{(12)}{(1a)(a2)}\frac{(2a)}{(2b)(ab)}}_{\sim \frac{1}{(k_{ta}^{(12)}k_{tb}^{(2a)})^2}} + \frac{(12)}{(1a)(a2)}\frac{(1a)}{(1b)(ab)}$$

- Each colour flow invariant under $\{\hat{n}_a \leftrightarrow \hat{n}_b; k_{ta}^{(12)} \leftrightarrow k_{tb}^{(ia)}\}$ (directions in the {12} frame), i.e.

$$\begin{bmatrix}\tilde{k}_a \\ \tilde{k}_b\end{bmatrix} = \mathbb{T}^{(i)}\begin{bmatrix}k_a \\ k_b\end{bmatrix} \qquad \mathbb{T}^{(1)} := \begin{bmatrix} 0 & \left(\frac{(1b)(a2)}{(12)(ab)}\right)^{-1/2} \\ \left(\frac{(1a)(b2)}{(12)(ab)}\right)^{1/2} & 0 \end{bmatrix}, \quad \mathbb{T}^{(2)} := \begin{bmatrix} 0 & \left(\frac{(1a)(b2)}{(12)(ab)}\right)^{-1/2} \\ \left(\frac{(1b)(a2)}{(12)(ab)}\right)^{1/2} & 0 \end{bmatrix}$$

- dLIPS measures for colour flows are mapped onto each other, while sources (observable) satisfy the same symmetry … Resum with evolution eq. ordered in dipole k_t

Second-order (planar) corrections to evolution kernel

$$Z_{12}[Q;\{u\}] = \mathbb{K}_{\text{int}}^{\text{RV+VV}}[Z[Q;u],u] + \mathbb{K}_{\text{int}}^{\text{RR}}[Z[Q;u],u] - \mathbb{K}_{\text{int}}^{\text{DC}}[Z[Q;u],u]$$

$$\mathbb{K}_{\text{int}}^{\text{DC}}[Z[Q;u],u] = \int [dk_a] \int [dk_b] \bar{\alpha}^2(k_{ta}) \Theta(Q - k_{ta}) \Theta(k_{ta} - k_{tb}) \frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})}$$
$$\times \Bigg[w_{12}^{(0)}(k_a) \left(w_{1a}^{(0)}(k_b) - \frac{1}{2} w_{12}^{(0)}(k_b) \right) Z_{1b}[k_{ta};\{u\}] Z_{ba}[k_{ta};\{u\}] Z_{a2}[k_{ta};\{u\}] u(k_a) u(k_b)$$
$$+ w_{12}^{(0)}(k_a) \left(w_{a2}^{(0)}(k_b) - \frac{1}{2} w_{12}^{(0)}(k_b) \right) Z_{1a}[k_{ta};\{u\}] Z_{ab}[k_{ta};\{u\}] Z_{b2}[k_{ta};\{u\}] u(k_a) u(k_b)$$
$$- w_{12}^{(0)}(k_a) \left(w_{1a}^{(0)}(k_b) + w_{a2}^{(0)}(k_b) - w_{12}^{(0)}(k_b) \right) Z_{1a}[k_{ta};\{u\}] Z_{a2}[k_{ta};\{u\}] u(k_a) \Bigg]$$

Perturbative insertion of double-real corrections

$$
\begin{aligned}
Z_{12}^{(1)}[Q;\{u\}] \simeq & \int [dk_a]\bar{\alpha}(k_{ta}) w_{12}^{(0)}(k_a)\,\frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \\
& \times \left(Z_{1a}^{(0)}[k_{ta};\{u\}] Z_{a2}^{(1)}[k_{ta};\{u\}] + Z_{1a}^{(1)}[k_{ta};\{u\}] Z_{a2}^{(0)}[k_{ta};\{u\}] \right) u(k_a)\Theta(Q-k_{ta}) \\
& + \int [dk_a] \int [dk_b]\, \bar{\alpha}^2(k_{t(ab)})\Theta(Q-k_{t(ab)})\Theta(k_{ta}-k'_{tb}) \frac{\Delta_{12}(Q)}{\Delta_{12}(k_{t(ab)})} \\
& \times \Big[\tilde{w}_{12}^{(0)}(k_b,k_a) Z_{1b}^{(0)}[k_{t(ab)};\{u\}] Z_{ba}^{(0)}[k_{t(ab)};\{u\}] Z_{a2}^{(0)}[k_{t(ab)};\{u\}] u(k_a)u(k_b) \\
& + \tilde{w}_{12}^{(0)}(k_a,k_b) Z_{1a}^{(0)}[k_{t(ab)};\{u\}] Z_{ab}^{(0)}[k_{t(ab)};\{u\}] Z_{b2}^{(0)}[k_{t(ab)};\{u\}] u(k_a)u(k_b) \\
& - \left(\tilde{w}_{12}^{(0)}(k_b,k_a) + \tilde{w}_{12}^{(0)}(k_a,k_b) \right) Z_{1(ab)}^{(0)}[k_{t(ab)};\{u\}] Z_{(ab)2}^{(0)}[k_{t(ab)};\{u\}] u(k_{(ab)}) \Big] \\
& - \int [dk_a] \int [dk_b] \bar{\alpha}^2(k_{ta})\Theta(Q-k_{ta})\Theta(k_{ta}-k_{tb}) \frac{\Delta_{12}(Q)}{\Delta_{12}(k_{ta})} \\
& \times \Big[w_{12}^{(0)}(k_a) w_{1a}^{(0)}(k_b) Z_{1b}^{(0)}[k_{ta};\{u\}] Z_{ba}^{(0)}[k_{ta};\{u\}] Z_{a2}^{(0)}[k_{ta};\{u\}] u(k_a)u(k_b) \\
& + w_{12}^{(0)}(k_a) w_{a2}^{(0)}(k_b) Z_{1a}^{(0)}[k_{ta};\{u\}] Z_{ab}^{(0)}[k_{ta};\{u\}] Z_{b2}^{(0)}[k_{ta};\{u\}] u(k_a)u(k_b) \\
& - w_{12}^{(0)}(k_a) \left(w_{1a}^{(0)}(k_b) + w_{a2}^{(0)}(k_b) \right) Z_{1a}^{(0)}[k_{ta};\{u\}] Z_{a2}^{(0)}[k_{ta};\{u\}] u(k_a) \Big]
\end{aligned}
$$

Fixed order expansion (full colour)

- Keep only terms up to NLL & extend to full colour (at fixed order only)

promote $(N_c)^n$ to correct Casimirs

$$\Sigma(v) \simeq 1 + \left(\frac{\alpha_s}{2\pi}\right)\left(\mathcal{H}_2^{(1)} - 4C_F \int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t) + \mathcal{H}_3^{(1)} \otimes \mathbb{1}\right)$$
$$- 4C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(k_t - Q)\left(K^{(1)} - 4\pi\beta_0 \ln\frac{k_t}{Q}\right)$$
$$+ 8C_F^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\int [dk] w_{12}^{(0)}(k)\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t)\right)^2$$
$$- 8C_F \left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk_a] \int [dk_b] \Big[C_A\left(\bar{w}_{12}^{(gg)}(k_a,k_b) + \bar{w}_{12}^{(gg)}(k_b,k_a)\right)$$
$$+ n_f\left(\bar{w}_{12}^{(q\bar{q})}(k_a,k_b) + \bar{w}_{12}^{(q\bar{q})}(k_b,k_a)\right)\Big]$$
$$\times \Theta(Q - k_{t(ab)})\Theta(k_{ta} - k_{tb})\left\{\Theta_{\rm out}(k_{(ab)})\left[\Theta_{\rm in}(k_a)\Theta_{\rm out}(k_b)\Theta(v(k_a)-v)\right.\right.$$
$$\left.\left. + \Theta_{\rm out}(k_a)\Theta_{\rm in}(k_b)\Theta(v(k_b)-v)\right] - \Theta_{\rm in}(k_{(ab)})\Theta_{\rm out}(k_a)\Theta_{\rm out}(k_b)\Theta(v(k_{(ab)})-v)\right\}$$
$$- 2\left(\frac{\alpha_s}{2\pi}\right)^2 \int [dk]\Theta_{\rm in}(k)\Theta(v(k)-v)\Theta(Q-k_t)$$
$$\times \left[2C_F\mathcal{H}_2^{(1)} w_{12}^{(0)}(k) + \mathcal{H}_3^{(1)} \otimes \left(C_A(w_{13}^{(0)}(k) + w_{32}^{(0)}(k)) + (2C_F - C_A)w_{12}(k)\right)\right].$$

add double soft fermionic current

add subl. colour 3-jet dipole