

Towards five loop calculations in QCD

Ben Ruijl*, Franz Herzog, Takahiro Ueda, Jos Vermaseren,
Andreas Vogt

*LCDS Leiden and Nikhef Amsterdam

Dec 06, 2016



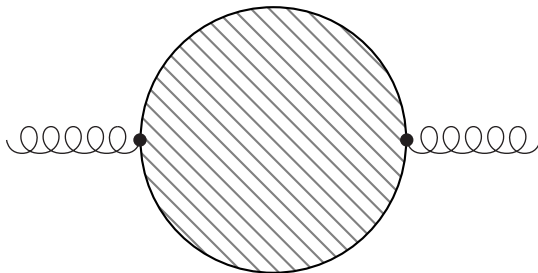
Universiteit
Leiden

Leiden Centre of Data Science



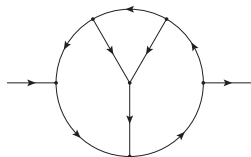
Forcer package

- We have built a program that calculates four-loop massless propagator diagrams
- Express diagrams as linear combinations of diagrams with fewer propagators



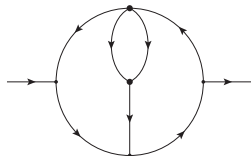
IBP identities

Through integration by parts (IBP) identities we find rules to remove lines:



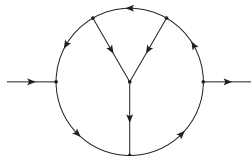
IBP identities

Through integration by parts (IBP) identities we find rules to remove lines:



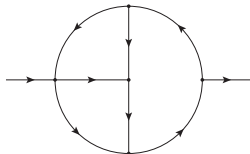
IBP identities

Through integration by parts (IBP) identities we find rules to remove lines:



IBP identities

Through integration by parts (IBP) identities we find rules to remove lines:



Identify substructures

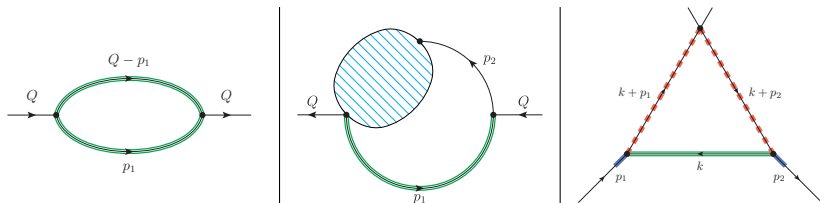


Figure: Guaranteed to remove a green or blue line

Triangle generalizations: diamonds [Ruijl,Ueda,Vermaseren '15]

Example reduction

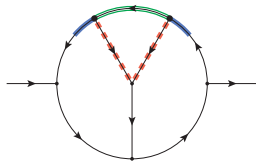


Figure: Reduction of the Benz diagram

Example reduction

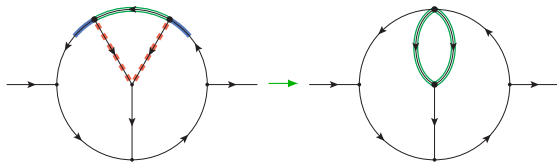


Figure: Reduction of the Benz diagram

Example reduction

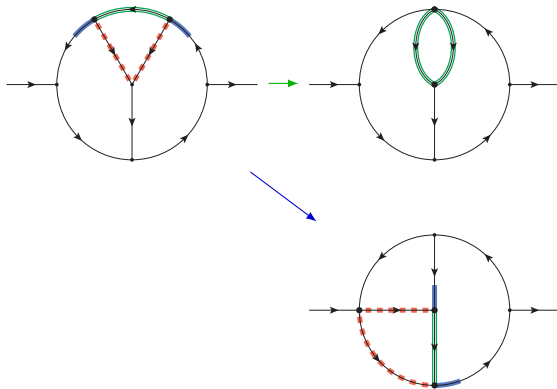


Figure: Reduction of the Benz diagram

Example reduction

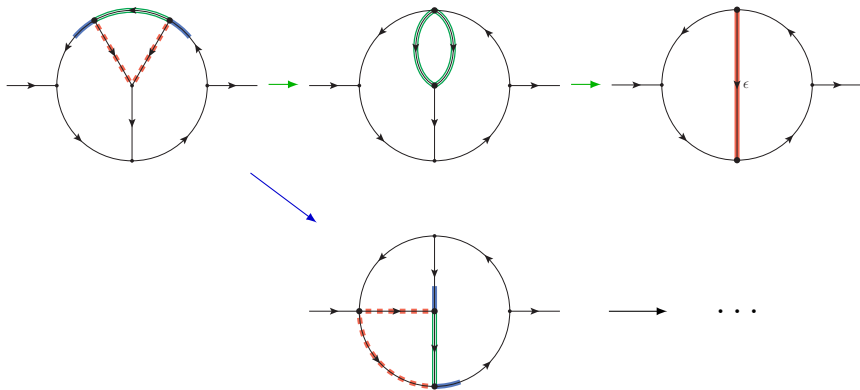


Figure: Reduction of the Benz diagram

3-loop reduction graph

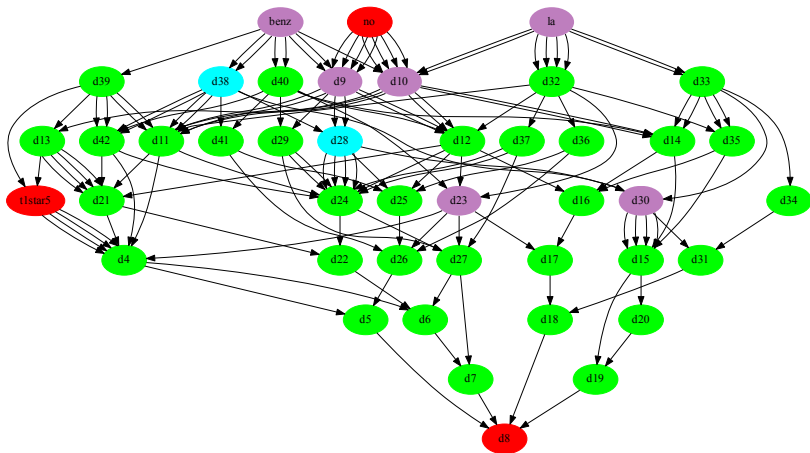


Figure: Reduction graph for 3 loop diagrams

4-loop reduction graph

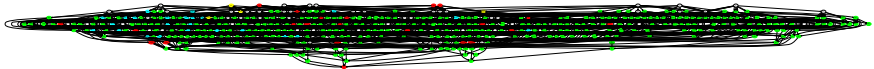


Figure: Reduction graph for 4 loop diagrams

4-loop reduction graph

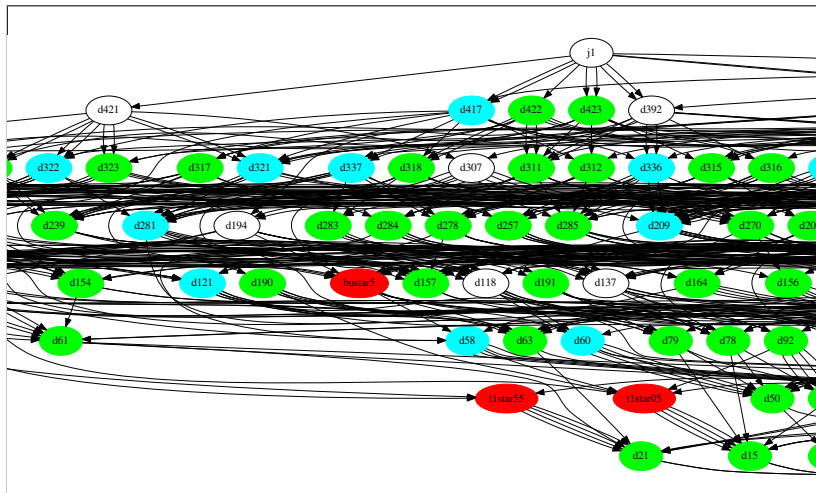
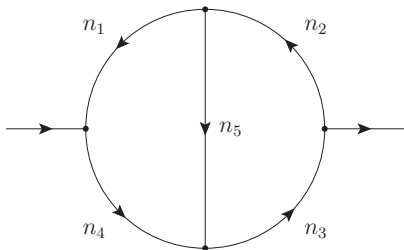


Figure: Part of reduction graph for 4 loops

Manual solutions

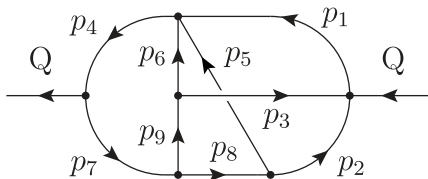
- 21 topologies do not have substructures
- Solve recursion relations parametrically
- Much faster than Laporta methods



$$I(n_1, n_2, n_3, n_4, n_5) = \\ n_1 I(n_1 - 1, n_2, n_3, n_4, n_5) \\ + n_2 I(n_1, n_2 + 1, n_3, n_4, n_5) + \dots$$

$$I(n_1, n_2, n_3, n_4, n_5) = \\ n_3 I(n_1, n_2, n_3 - 1, n_4, n_5) \\ + n_3 I(n_1, n_2 + 1, n_3, n_4, n_5) + \dots$$

Public enemy #1



- 9 propagators that should go to 1
- 5 irreducible dot products that should go to 0
- Took 3 months to find solution!

Reduction rule

```

id,ifmatch->bubul,
  Z(n1?pos_,n2?pos_,n3?pos_,n4?pos_,n5?pos_,n6?pos_,n7?pos_,
    n8?pos_,n9?pos_,n10?neg0_,n11?neg0_,n12?neg0_,n13?neg0_,n14?neg_)
    = -rat(1,-2*ep-2*n1-n3-n6-n12-n14+4)*(
+Z(-1+n1,-1+n2,n3,n4,1+n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-n5,1)
+Z(-1+n1,1+n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(n2,1)
+Z(-1+n1,1+n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(-n2,1)
+Z(-1+n1,n2,1+n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(-n3,1)
+Z(-1+n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,1+n12,n13,1+n14)*rat(-n12,1)
+Z(-1+n1,n2,n3,n4,1+n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(n5,1)
+Z(-1+n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,1+n12,n13,1+n14)*rat(2*n12,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,2+n14)*rat(2*n14+2,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,1+n10,-1+n11,n12,n13,1+n14)*rat(-n10,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,n14)*rat(-n12,1)
+Z(-1+n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-n2+n5,1)
+Z(n1,-1+n2,-1+n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,1+n14)*rat(n12,1)
+Z(n1,-1+n2,-1+n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,2+n14)*rat(1+n14,1)
+Z(n1,-1+n2,1+n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(n3,1)

```

+Z(n1, -1+n2, n3, -1+n4, n5, n6, n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)*rat(n11, 1)
+Z(n1, -1+n2, n3, n4, -1+n5, n6, n7, n8, n9, n10, n11, 1+n12, n13, 1+n14)*rat(n12, 1)
+Z(n1, -1+n2, n3, n4, n5, -1+n6, n7, n8, n9, n10, n11, n12, n13, 2+n14)*rat(-n14-1, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, 1+n8, n9, n10, n11, n12, n13, 1+n14)*rat(2*n8, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)*rat(-n11, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, -1+n7, n8, n9, n10, n11, n12, 1+n13, 1+n14)*rat(-n13, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, 1+n12, n13, 1+n14)*rat(-2*n12, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, 1+n13, 1+n14)*rat(n13, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 2+n14)*rat(-2*n14-2, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, 1+n8, -1+n9, n10, n11, n12, n13, 1+n14)*rat(-n8, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, -1+n9, n10, n11, 1+n12, n13, 1+n14)*rat(-2*n12, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, 1+n10, -1+n11, n12, n13, 1+n14)*rat(n10, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, 1+n10, n11, n12, n13, 1+n14)*rat(-2*n10, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, n10, 1+n11, n12, n13, 1+n14)*rat(-n11, 1)
+Z(n1, -1+n2, n3, n4, n5, n6, n7, n8, n9, n10, n11, 1+n12, n13, n14)*rat(-n12, 1)
+Z(n1, 1+n2, n3, n4, -1+n5, n6, n7, n8, n9, -1+n10, n11, n12, n13, 1+n14)*rat(-n2, 1)
+Z(n1, 1+n2, n3, n4, -1+n5, n6, n7, n8, n9, n10, n11, n12, n13, 1+n14)*rat(-n2, 1)
+Z(n1, 1+n2, n3, n4, n5, n6, n7, -1+n8, n9, -1+n10, n11, n12, n13, 1+n14)*rat(2*n2, 1)
+Z(n1, 1+n2, n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 1+n14)*rat(n2, 1)
+Z(n1, n2, -1+n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, 1+n12, n13, 1+n14)*rat(-n12, 1)
+Z(n1, n2, -1+n3, n4, n5, n6, n7, -1+n8, n9, n10, n11, n12, n13, 2+n14)*rat(-n14-1, 1)
+Z(n1, n2, 1+n3, n4, n5, n6, n7, n8, -1+n9, -1+n10, n11, n12, n13, 1+n14)*rat(n3, 1)

+Z(n1,n2,n3,n4,-1+n5,n6,-1+n7,n8,n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,2+n14)*rat(1+n14,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,-1+n10,n11,1+n12,n13,1+n14)*rat(n12,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,-1+n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,1+n12,n13,1+n14)*rat(n12,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(n1,n2,n3,n4,-1+n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-n2+n8-n13,1)
+Z(n1,n2,n3,n4,n5,-1+n6,n7,-1+n8,n9,n10,n11,n12,n13,2+n14)*rat(1+n14,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,-1+n8,n9,n10,1+n11,n12,n13,1+n14)*rat(n11,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,-1+n8,n9,n10,n11,n12,1+n13,1+n14)*rat(-2*n13,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,-1+n10,1+n11,n12,n13,1+n14)*rat(n11,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,-1+n12,n13,2+n14)*rat(1+n14,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(n1,n2,n3,n4,n5,n6,-1+n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(-2*ep-2*n4-1,1)
+Z(n1,n2,n3,n4,n5,n6,1+n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(-n7,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,-1+n9,n10,n11,n12,n13,2+n14)*rat(-2*n14-2,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,-1+n10,1+n11,n12,n13,1+n14)*rat(-n11,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,-1+n10,n11,1+n12,n13,1+n14)*rat(-2*n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,-1+n10,n11,n12,n13,2+n14)*rat(-2*n14-2,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,1+n10,n11,n12,n13,1+n14)*rat(2*n10,1)

```
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,-1+n11,n12,1+n13,1+n14)*rat(-2*n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,1+n11,n12,n13,1+n14)*rat(n11,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,-1+n12,n13,2+n14)*rat(-n14-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,1+n12,n13,1+n14)*rat(-2*n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,1+n13,1+n14)*rat(-3*n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,1+n14)*rat(10*ep+2*n1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,-1+n8,n9,n10,n11,n12,n13,2+n14)*rat(-2*n14-2,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,-1+n10,n11,1+n12,n13,1+n14)*rat(-n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,-1+n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,-1+n11,n12,n13,2+n14)*rat(-n14-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,1+n13,1+n14)*rat(-n13,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,-1+n9,n10,n11,n12,n13,1+n14)*rat(-4*ep-2*n1-n3,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,-1+n10,n11,n12,n13,1+n14)*rat(-n5+1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,1+n10,-1+n11,n12,n13,1+n14)*rat(n10,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,-1+n11,n12,n13,1+n14)*rat(2*ep+n5+2*n8
+n9+n10+n11-5,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,1+n12,n13,n14)*rat(n12,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,-1+n13,1+n14)*rat(-2*ep-n5-2*n8
-n9-n11-n14+3,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9,n10,n11,n12,n13,1+n14)*rat(2*ep+n2+n7+
2*n8+n9+n11+n14-4,1)
);
```

First results and benchmarks

- Reproduced 4-loop QCD β -function [Ritbergen, Vermaseren, Larin '97; Czakon '04]
- All ξ and ϵ 4-loop solution for QCD propagators and vertices [new!]

β_3 no gauge	10 minutes
β_3 1 gauge	38 minutes
β_3 all gauge	8.5 hours
no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)	42 minutes

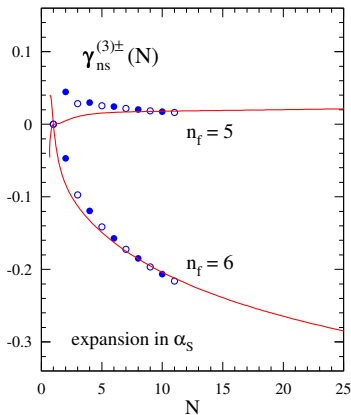
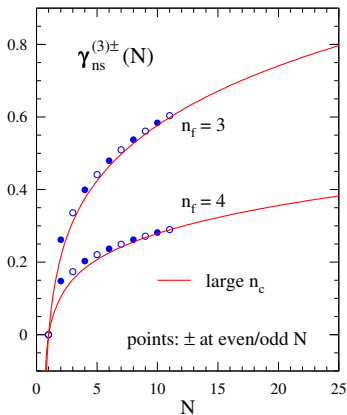
Table: Benchmark on 24 core machine

4-loop splitting functions

- Can we calculate the 4-loop corrections to the PDF evolution?
- Calculate Mellin moments N for as many N as possible
- Each $N \rightarrow N + 2$ increases weight by 4
- 4-loop time: more than 1000 times 3-loop time

$$\gamma_{ij} = -P_{ij} = \sum_n \left(\frac{\alpha_s}{4\pi} \right)^n P_{ij}^{(n-1)} \quad i, j = q, g$$

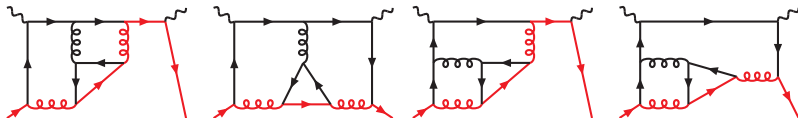
Non-singlet (NS): combinations of γ_{qq}

N3LO corrections to γ_{NS} 

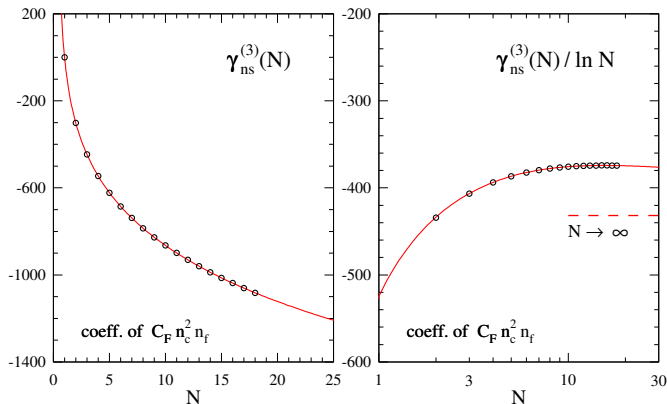
- $N = 2, 3, 4$ in agreement with [Baikov,Chetyrkin '06, Velizhanin '13,'14]
- New results from $N = 5$ to $N = 11$ [new; Moch,RUVV]

All- N results for n_f^3 to n_f^0

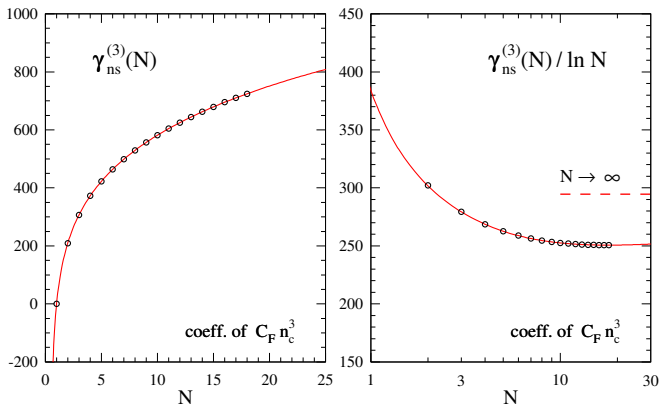
All 3-loop graphs (with insertion on gluon line) for n_f^2 in γ_{qq} :



- Simple topologies for forcer \rightarrow high moments
- Analytic form in N reconstructed with LLL-method
[Velizhanin '12; Moch, Vermaseren, Vogt '14]
- Singlet splitting function matrix is now known at leading n_f
[new; Davies, RUVV]
- n_f^1 and n_f^0 now known for γ_{NS} in large n_c limit using OPE [new; Moch, RUVV]
- The hardest topologies drop out in large n_c limit

n_f^1 coefficient in large n_c limit

- Reconstruction using 18 moments, checked with $N = 19$
- Confirmed result ($N \rightarrow \infty$) from [Henn,Steinhauser,Smirnov², '15]
- Agrees with predictions from small- x resummation

n_f^0 coefficient in large n_c limit

- Reconstruction using 18 moments, checked with $N = 19$
- Agrees with predictions of structural relations
- New term in cusp anomalous dimension!

New cusp term: n_f^0 in large n_c limit

$$\begin{aligned}\gamma_{\text{cusp}}^{(3)} = & +n_c^3 C_F \left(+\frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_3 \zeta_2 \right. \\ & \left. - 352 \zeta_5 - 32 \zeta_3^2 - 876 \zeta_6 \right) \\ & + n_c^2 C_F n_f \left(-\frac{39883}{81} + \frac{26692}{81} \zeta_2 - \frac{16252}{27} \zeta_3 - \frac{440}{3} \zeta_4 + \frac{256}{3} \zeta_3 \zeta_2 + 224 \zeta_5 \right) \\ & + n_c C_F n_f^2 \left(\frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) \\ & + C_F n_f^3 \left(-\frac{32}{81} + \frac{64}{27} \zeta_3 \right) \\ & + \dots\end{aligned}$$

Five loops

- Can we go to five loops?
- Recompute 5-loop QCD β -function [Baikov, Chetyrkin, Kühn '16] with general colour group and gauge? [see also: Luthe, Maier, Marquard, Schröder '16]
- 5-loop Forcer is hard:
 - Unknown master integrals
 - 20 parameter problem
 - More than 200 manual reductions!
 - However: 30% of topologies factorize

Five loops

- We are interested in anomalous dimensions
- Thus, we only need the UV pole part
- Can we IR rearrange to create a *carpet* diagram?

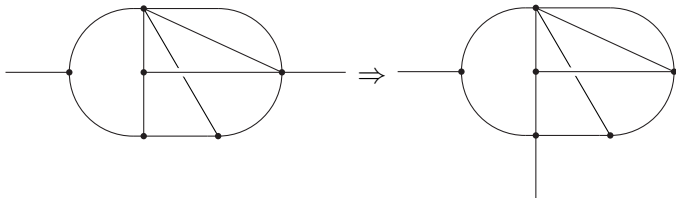


Figure: The graphs have the **same** UV poles

The R operation

- R is a renormalization operator
- BPHZ renormalization scheme
[Bogoliubov,Parasiuk,Hepp,Zimmermann]
- R can be expressed as a recursive subtraction operator
[Chetyrkin,Smirnov,Larin]
- The solution is Zimmermann's forest formula
- We are going to exploit that R can isolate the UV poles

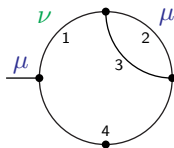
Assumptions

External momenta are massive, as collinear divergences are not supported.

Some applications for R and R^*

- $Z, \tau \rightarrow \text{Hadrons} / Z\text{-decay}$: QCD at N4LO (Global R^*)
[Baikov,Chetyrkin,Kühn,'12]
- Anomalous dimensions
 - QCD
 - 5-loop β function (Global R^*) [Baikov,Chetyrkin,Kühn,'16]
 - 5-loop quark mass and field anomalous dimension (Global R^*)
[Baikov,Chetyrkin,Kühn,'14]
 - ϕ^4
 - 6-loop wavefunction (Global & Local R^*)
[Batkovich,Chetyrkin,Kompaniets,'16]
 - 6-loop β function (Local R^*) [Kompaniets, Panzer,'16]
- Decay rates: QCD corrections
 - $H \rightarrow b\bar{b}$ N4LO (Global & Local R^*) [Baikov,Chetyrkin,Kühn,'05]
- g-2: QED corrections
 - 5 loop (Local R^* (?), even numerical)
[Kinoshita,Aoyama,Hayakawa,Kinoshita,Nio,'14]
- Automation (for ϕ^4) [Batkovich,Kompaniets,'14]

Superficial UV divergence



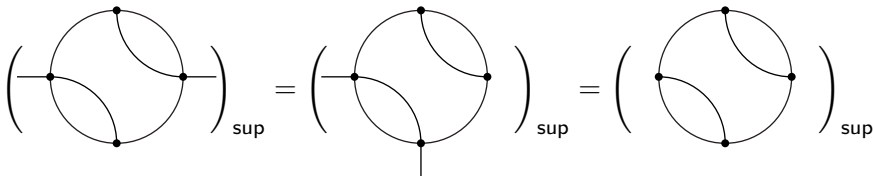
$$= \int d^D p_1 \int d^D p_2 \frac{Q \cdot p_2 p_1^\nu}{p_1^2 p_2^2 p_3^2 p_4^4}$$

- Superficial UV divergence: all loop momenta $\rightarrow \infty$
- Get degree of divergence through power counting:
 - Each loop contributes +4 due to the measure
 - $8 + 1 + 1 - 2 - 2 - 2 - 4 = 0$

Logarithmically divergent integrals

Infrared rearrangement

UV pole parts of log integrals do not depend on kinematics or masses



Linearly divergent integrals

$$\left(\text{circle with two external lines} \right)_{\text{sup}} \propto Q^\mu \neq \left(\text{circle with two external lines} \right)_{\text{sup}}$$

Taylor expand in external momenta:

$$\begin{aligned} \left(\text{circle with two external lines} \right)_{\text{sup}} &= \left(\text{circle with two external lines} + Q^\alpha \partial_{Q^\alpha} \left(\text{circle with two external lines} \right) \Big|_{Q=0} + \dots \right)_{\text{sup}} \\ &= \left(\text{circle with two external lines} - 2Q^\alpha \left(\text{circle with two external lines and one internal line labeled } \alpha \right) + \dots \right)_{\text{sup}} \end{aligned}$$

UV Taylor expansion

$$\left(\text{Diagram} \right)_{\text{sup}} = \left(\underbrace{\text{Diagram}}_{\text{Lower order}} - 2Q^\alpha \text{Diagram} + \underbrace{\dots}_{\text{Higher order}} \right)_{\text{sup}}$$

The diagram on the left is a circle with two external lines on the left and a scale parameter μ above it. The diagram in the middle is a circle with two external lines on the left and a scale parameter μ above it, with a bracket underneath labeled "Lower order". The diagram on the right is a circle with two external lines on the left and a scale parameter μ above it, with a scale parameter α below it. The ellipsis is bracketed underneath and labeled "Higher order".

- Higher orders do not have a superficial divergence and are 0
- Lower orders are $> \log$ massless tadpoles and are 0 because there is no scale

Thus:

$$\left(\text{Diagram} \right)_{\text{sup}} = -2Q^\alpha \left(\text{Diagram} \right)_{\text{sup}}$$

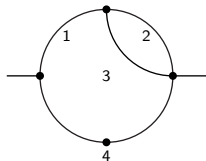
The diagram on the left is a circle with two external lines on the left and a scale parameter μ above it. The diagram on the right is a circle with two external lines on the left and a scale parameter μ above it, with a scale parameter α below it.

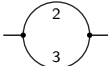
Subdivergences

- Subdivergence: singular behaviour when **some** loop momenta go to infinity (or zero)
- No subdivergences: superficial divergence is pole part
- Introduce pole operator: $K(\sum_{i=-\infty}^{\infty} c_i \epsilon^i) = \sum_{i=-\infty}^{-1} c_i \epsilon^i$

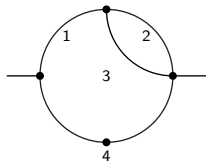
$$\left(\text{circle with two external lines} \right)_{\text{sup}} = K \left(\text{circle with two external lines} \right) = \frac{1}{\epsilon}$$

Subdivergences

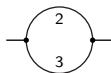


- Has log subdivergence: 
- Superficial divergence is pole part minus subdivergences

Subdivergences



- Has log subdivergence:

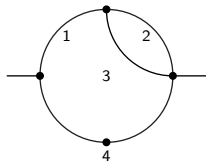


- Superficial divergence is pole part minus subdivergences

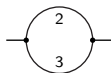
$$\left(\text{Diagram} \right)_{\text{sup}} = K \left(\text{Diagram} \right) - K \left(\text{Diagram} \right) \left(\text{Diagram} \right)$$

The equation shows the subtraction of subdivergences from the superficial divergence of a four-loop diagram. The left side is the four-loop diagram in parentheses with a subscript 'sup'. The right side is the product of the four-loop diagram in parentheses and the two-loop diagram in parentheses, with a minus sign between them.

Subdivergences



- Has log subdivergence:



- Superficial divergence is pole part minus subdivergences

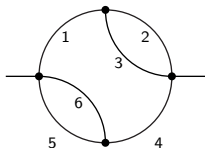
$$\left(\text{Diagram} \right)_{\text{sup}} = K \left(\text{Diagram} \right) - K \left(\text{Diagram} \right) \left(\text{Diagram} \right)$$

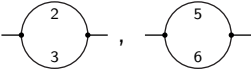
The equation shows the subtraction of subdivergences from the superficial divergence. The first term is the four-loop diagram in parentheses with a subscript 'sup'. The second term is the same four-loop diagram in parentheses multiplied by K. The third term is a two-loop diagram in parentheses multiplied by K, followed by a one-loop diagram in parentheses. Green arrows point from the text 'Pole of subdivergence' to the two-loop diagram and from 'Remaining diagram' to the one-loop diagram.

Pole of subdivergence

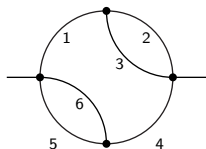
Remaining diagram

Multiple subdivergences



- Has log subdivergences: 
- Loop momenta can **independently** go to infinity

Multiple subdivergences



- Has log subdivergences: ,
- Loop momenta can **independently** go to infinity

$$\left(\text{Diagram} \right)_{\text{sup}} = K \left(\text{Diagram} \right) - K \left(\text{Diagram} \right) \text{Diagram} - K \left(\text{Diagram} \right) \text{Diagram} + K \left(\text{Diagram} \right) K \left(\text{Diagram} \right) \text{Diagram}$$

The equation shows the subtraction of subdivergences from a four-loop diagram. The left side is the four-loop diagram with a 'sup' label. The right side is a sum of terms: the original four-loop diagram, minus two terms representing the subtraction of subdivergences (each a two-loop diagram multiplied by the four-loop diagram), plus a term representing the subtraction of the product of two subdivergences (two two-loop diagrams multiplied together and then multiplied by the four-loop diagram).

Overlapping subdivergences (I)

$$\begin{aligned}
 & \left(\text{Diagram 1} \right)_{\text{sup}} = K \left(\text{Diagram 2} \right) - K \left(\text{Diagram 3} \right) - K \left(\text{Diagram 4} \right) \\
 & \quad - K \left(\text{Diagram 5} \right) - \left(\text{Diagram 6} \right)_{\text{sup}} - \left(\text{Diagram 7} \right) \\
 & \quad + K \left(\text{Diagram 8} \right) K \left(\text{Diagram 9} \right) - \left(\text{Diagram 10} \right)
 \end{aligned}$$

The diagrams are:

- Diagram 1: A circle with two external lines on the left and right. Internal lines are labeled 1, 2, 3, 4, 5, 6. A subdivergence is shown as a shaded region bounded by lines 2, 3, 4, 5.
- Diagram 2: Same as Diagram 1, but with the subdivergence region shaded.
- Diagram 3: A circle with two external lines on the left and right. Internal lines are labeled 2, 3. A subdivergence is shown as a shaded region bounded by lines 2, 3.
- Diagram 4: A circle with two external lines on the left and right. Internal lines are labeled 1, 4, 5, 6. A subdivergence is shown as a shaded region bounded by lines 1, 4, 5, 6.
- Diagram 5: A circle with two external lines on the left and right. Internal lines are labeled 4, 5, 6. A subdivergence is shown as a shaded region bounded by lines 4, 5, 6.
- Diagram 6: Two circles sharing a vertex. The left circle has internal lines 2, 3 and the right circle has internal lines 4, 5. A subdivergence is shown as a shaded region bounded by lines 2, 3, 4, 5.
- Diagram 7: A circle with two external lines on the left and right. Internal lines are labeled 1, 6. A subdivergence is shown as a shaded region bounded by lines 1, 6.
- Diagram 8: A circle with two external lines on the left and right. Internal lines are labeled 2, 3. A subdivergence is shown as a shaded region bounded by lines 2, 3.
- Diagram 9: A circle with two external lines on the left and right. Internal lines are labeled 4, 5. A subdivergence is shown as a shaded region bounded by lines 4, 5.
- Diagram 10: A circle with two external lines on the left and right. Internal lines are labeled 1, 6. A subdivergence is shown as a shaded region bounded by lines 1, 6.

Overlapping subdivergences (I)

$$\begin{aligned}
 & \left(\text{Diagram 1} \right)_{\text{sup}} = K \left(\text{Diagram 2} \right) - K \left(\text{Diagram 3} \right) - K \left(\text{Diagram 4} \right) \\
 & \quad - K \left(\text{Diagram 5} \right) - \left(\text{Diagram 6} \right)_{\text{sup}} - \text{Diagram 7} \\
 & \quad + \cancel{K \left(\text{Diagram 8} \right)} - \cancel{K \left(\text{Diagram 9} \right)} - \cancel{K \left(\text{Diagram 10} \right)}
 \end{aligned}$$

Diagram 1: A circle with vertices 1, 2, 3, 4, 5, 6. A loop is formed by vertices 2, 3, 4, 5.

Diagram 2: A circle with vertices 1, 2, 3, 4, 5, 6. A loop is formed by vertices 2, 3, 4, 5.

Diagram 3: A circle with vertices 2, 3.

Diagram 4: A circle with vertices 1, 4, 5, 6. A loop is formed by vertices 1, 4, 5, 6.

Diagram 5: A circle with vertices 4, 5, 6. A loop is formed by vertices 1, 2, 3, 6.

Diagram 6: Two circles sharing a vertex. The left circle has vertices 2, 3, 5. The right circle has vertices 4, 5, 6.

Diagram 7: A circle with vertices 1, 6.

Diagram 8: A circle with vertices 2, 3, 5.

Diagram 9: A circle with vertices 4, 5, 6.

Diagram 10: A circle with vertices 1, 4, 5, 6.

A red arrow points from the text "Undersubtracting!" to the last three terms of the equation, which are crossed out with red lines.

Overlapping subdivergences (II)

$$\begin{aligned}
 \left(\text{Diagram 1} \right)_{\text{sup}} &= K \left(\text{Diagram 2} \right) - K \left(\text{Diagram 3} \right) - K \left(\text{Diagram 4} \right) \\
 &\quad - K \left(\text{Diagram 5} \right) \\
 &\quad - K \left(\text{Diagram 6} - K \left(\text{Diagram 7} \right) - K \left(\text{Diagram 8} \right) \right) \\
 &\quad - K \left(\text{Diagram 9} \right)
 \end{aligned}$$

The diagrams are:

- Diagram 1:** A circle with two external lines on the left and right. Internal lines are labeled 1, 2, 3, 4, 5, 6. A subdivergence is shown as a shaded region bounded by lines 2, 3, 4, and 5.
- Diagram 2:** Similar to Diagram 1, but the subdivergence region is not shaded.
- Diagram 3:** A circle with two external lines on the left and right. Internal lines are labeled 2 and 3.
- Diagram 4:** A circle with two external lines on the left and right. Internal lines are labeled 1, 4, 5, and 6. A subdivergence is shown as a shaded region bounded by lines 1, 4, 5, and 6.
- Diagram 5:** A circle with two external lines on the left and right. Internal lines are labeled 4 and 5. A subdivergence is shown as a shaded region bounded by lines 1, 2, 3, and 6.
- Diagram 6:** Two circles connected by a line. The left circle has internal lines 2 and 3; the right circle has internal lines 4 and 5.
- Diagram 7:** A circle with two external lines on the left and right. Internal lines are labeled 2 and 3.
- Diagram 8:** A circle with two external lines on the left and right. Internal lines are labeled 4 and 5.
- Diagram 9:** A circle with two external lines on the left and right. Internal lines are labeled 4 and 5.

Overlapping subdivergences (II)

$$\begin{aligned}
 & \left(\text{Diagram 1} \right)_{\text{sup}} = K \left(\text{Diagram 2} \right) - K \left(\text{Diagram 3} \right) - K \left(\text{Diagram 4} \right) \\
 & \quad - K \left(\text{Diagram 5} \right) - K \left(\text{Diagram 6} \right) - K \left(\text{Diagram 7} \right) \\
 & \quad - K \left(\text{Diagram 8} \right) - K \left(\text{Diagram 9} \right) - K \left(\text{Diagram 10} \right)
 \end{aligned}$$

Nested structure



\bar{R} rule

- Overlapping divergences give rise to nested/recursive structures
- Enter the \bar{R} operation:

$$G_{\text{sup}} \equiv K\bar{R}G = K \sum_{S \in \bar{W}(G)} \left[\prod_{\gamma \in S} -K\bar{R}\gamma \right] * G/S$$

- $W(G)$ is the set of all sets of non-overlapping UV subdiagrams
- G/S is the remaining (quotient) diagram
- γ are symmetrized massless vacuum diagrams [new]

Tensor reduction

$$K\bar{R}\left(\text{Diagram 1}\right) = K\left(\text{Diagram 2}\right) - K\left(\text{Diagram 3}\right) - K\left(\text{Diagram 4}\right)$$

The equation shows the reduction of a tensor integral. The left-hand side is $K\bar{R}$ applied to a diagram with a circle and a diagonal line, with external indices $\mu\nu$. The right-hand side is the difference of three diagrams: the same diagram as on the left, a diagram with a circle and a horizontal line, and a diagram with a circle and a vertical line, both with external indices $\mu\nu$.

Tensor reduce:

$$K\left(\text{Diagram 3}\right) = g^{\mu\nu} K\left(\frac{1}{D} \text{Diagram 4}\right)$$

The equation shows the tensor reduction of the diagram with a circle and a vertical line. It is equal to the metric tensor $g^{\mu\nu}$ multiplied by the scalar integral K of a diagram with a circle and a horizontal line, scaled by $\frac{1}{D}$.

Commutativity issues

- Contractions do not commute with R
- ϵ and thus D does not commute with K
- \rightarrow Feynman rules do not commute with K nor R

$$\begin{aligned}
 g^{\mu\nu} K \bar{R} \left(\text{Diagram 1} \right) &= K \left(\text{Diagram 2} \right) \\
 &\quad - D K \left(\frac{1}{D} \text{Diagram 3} \right) \text{Diagram 4} \\
 &\neq K \bar{R} \left(\text{Diagram 1} \right)
 \end{aligned}$$

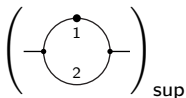
Diagram 1: A circle with two external lines on the left and right. A curved line connects the top and right sides of the circle, with a dot on the top arc labeled $\mu\nu$.
 Diagram 2: A circle with two external lines on the left and right. A curved line connects the top and right sides of the circle.
 Diagram 3: A circle with two external lines on the left and right.
 Diagram 4: A circle with two external lines on the left and right.

IR divergences

- What if loop momenta go to 0?
- All line combinations that go soft could be IR
- Not all lines with loop momentum are involved

IR divergences

- What if loop momenta go to 0?
- All line combinations that go soft could be IR
- Not all lines with loop momentum are involved



IR divergences

- What if loop momenta go to 0?
- All line combinations that go soft could be IR
- Not all lines with loop momentum are involved

$$\left(\text{circle with lines 1 and 2} \right)_{\text{sup}} = \left(\text{circle with lines } p \text{ and } Q-p \right)_{\text{sup}}$$

IR divergences

- What if loop momenta go to 0?
- All line combinations that go soft could be IR
- Not all lines with loop momentum are involved

$$\left(\text{circle with lines 1 and 2} \right)_{\text{sup}} = \left(\text{circle with lines } p \text{ and } Q-p \right)_{\text{sup}} = K \left(\text{circle with lines 1 and 2} \right) - \left(\text{circle with lines } 1 \text{ and } 0 \right) \text{---} \text{arc with line 2}$$

External to IR

IR divergences

Recipe:

- Remove IR lines from diagram
- Set the IR momenta to 0 in remaining diagram
- Actual IR graph has external vertices shrunk
- Degree of IR divergence: minus UV divergence of IR graph

$$\begin{aligned}
 \text{Diagram 1} &= \int \frac{d^D p}{p^4} = \text{Diagram 2} \\
 \text{Diagram 3} &= \int \frac{d^D p_1 d^D p_2}{p_1^4 p_2^2 (p_1 - p_2)^2} = \text{Diagram 4}
 \end{aligned}$$

Diagram 1: A vertical line with two external vertices (circles) at the ends and a black dot in the middle.

 Diagram 2: A circle with two external vertices (circles) at the top and bottom.

 Diagram 3: A horizontal line with two external vertices (circles) at the ends, a black dot in the middle, and a small loop attached to the right side.

 Diagram 4: A circle with two external vertices (circles) at the top and bottom, and a horizontal line segment passing through the center.

\bar{R}^* rule

- \bar{R}^* combines UV and IR subtractions [Chetyrkin,Smirnov]

$$K\bar{R}^*G = K \sum_{\substack{S \in \bar{W}(G) \\ S' \in \bar{W}'(G) \\ S \cap S' = \emptyset}} \left[\prod_{\gamma' \in S'} -K\underline{R}^* \gamma' \right] * \left[\prod_{\gamma \in S} -K\bar{R}^* \gamma \right] * G/S \setminus S'$$

- $W'(G)$ is the set of all non-overlapping IR subdiagrams
- $K\underline{R}^*$ is the superficial IR divergence [new]

Superficial IR

- How to compute the superficial IR divergence?

$$K(G) = K\bar{R}^*(G) + K\underline{R}^*(G) + CT(G)$$

- $CT(G)$ are the subdivergence counter terms of G
- $K(G) = 0$ since IR graphs are massless tadpoles

$$K\underline{R}^*(G) = -K\bar{R}^*(G) - CT(G)$$

Superficial IR: examples

Without subdivergences:

$$K\underline{R}^* \left(\begin{array}{c} \circ \\ | \\ \bullet \\ | \\ \circ \end{array} \right) = -K\bar{R}^* \left(\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right) = -\frac{1}{\epsilon}$$

With subdivergence:

$$K\underline{R}^* \left(\begin{array}{c} \circ \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \circ \end{array} \right) = -K\bar{R}^* \left(\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right) - K\underline{R}^* \left(\begin{array}{c} \circ \\ | \\ \bullet \\ | \\ \circ \end{array} \right) K \left(\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \right)$$

Linear IR divergences

$$\left(\text{Diagram with loop } p \text{ and } Q-p \right) = K \left(\text{Diagram with loop } \mu \right) - \left(\text{Diagram with loop } \mu \right) \text{ --- } \left(\text{Diagram with loop } Q-p \right)$$

The diagrammatic equation shows the expansion of a loop diagram with momenta p and $Q-p$ into a loop diagram with momenta μ and a diagram with a loop μ and a loop $Q-p$.

- Taylor expand the **remaining diagram** around $p = 0$
- Again, only the linear term survives

Linear IR divergences

$$\left(\text{Diagram with loop } p \text{ and } Q-p \right) = K \left(\text{Diagram with loop } \mu \right) - \left(\text{Diagram with vertical line } \mu \right) \text{---} \left(\text{Diagram with loop } Q-p \right)$$

- Taylor expand the **remaining diagram** around $p = 0$
- Again, only the linear term survives

$$\left(\text{Diagram with vertical line } \mu \right) \text{---} \left(\text{Diagram with loop } Q-p \right) = \left(\text{Diagram with vertical line } \mu\alpha \right) \partial_{p^\alpha} \left(\text{Diagram with loop } Q-p \right) \Big|_{p=0} = -2 \left(\text{Diagram with vertical line } \mu\alpha \right) \text{---} \alpha$$

Tricky interactions

$$K\bar{R}^* \left(\text{Diagram 1} \right) = -K\bar{R}^* \left(\text{Diagram 2} \right) + \left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right)$$

The equation shows a relationship between four Feynman diagrams. Diagram 1 is a circle with a horizontal line through its center, two dots on the line, and two dots at the bottom. Diagram 2 is similar but with two dots at the top. Diagram 3 is a vertical line with four dots. Diagram 4 is a simple circle with two dots on the horizontal line.

- Higher order UV may create vectors in remaining diagram
- IR Taylor should be sensitive to those!

Tricky interactions

$$K\bar{R}^* \left(\text{Diagram 1} \right) = -K\bar{R}^* \left(\text{Diagram 2} \right) + \left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right)$$

Diagram 1: A circle with a horizontal line through its center. The top and bottom arcs are labeled μ . There are two black dots on the horizontal line, one on each side of the center.

Diagram 2: A circle with a horizontal line through its center. The top and bottom arcs are labeled μ . There are two black dots on the horizontal line, one on each side of the center.

Diagram 3: A vertical line with two open circles at the top and bottom. A black dot is in the middle, and the label μ is to its right.

Diagram 4: A circle with two black dots on its left and right sides. The top arc is labeled μ .

- Higher order UV may create vectors in remaining diagram
- IR Taylor should be sensitive to those!

$$\left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right) = -2 \left(\text{Diagram 3} \right) K \left(\text{Diagram 5} \right) p^\alpha$$

Diagram 5: A circle with two black dots on its left and right sides. The top arc is labeled μ . There is a black dot at the bottom of the circle, and the label α is below it.

(UV Taylor)

Tricky interactions

$$K\bar{R}^* \left(\text{Diagram 1} \right) = -K\bar{R}^* \left(\text{Diagram 2} \right) + \left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right)$$

Diagram 1: A circle with a horizontal line through its center. The top and bottom vertices of the circle are labeled μ . There are two black dots on the bottom line, one on each side of the center.

Diagram 2: A circle with a horizontal line through its center. The top and bottom vertices of the circle are labeled μ . There are two black dots on the bottom line, one on each side of the center.

Diagram 3: A vertical line with two open circles at the top and bottom. A black dot is in the middle, with a μ label to its right.

Diagram 4: A circle with two vertices on the left and right sides. The top vertex is labeled μ .

- Higher order UV may create vectors in remaining diagram
- IR Taylor should be sensitive to those!

$$\begin{aligned} \left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right) &= -2 \left(\text{Diagram 3} \right) K \left(\text{Diagram 5} \right) p^\alpha && \text{(UV Taylor)} \\ &= -2g^{\alpha\beta} \left(\text{Diagram 6} \right) K \left(\text{Diagram 7} \right) && \text{(IR Taylor)} \end{aligned}$$

Diagram 5: A circle with two vertices on the left and right sides. The top vertex is labeled μ . The bottom vertex is labeled α . There is a black dot on the bottom line.

Diagram 6: A vertical line with two open circles at the top and bottom. Two black dots are in the middle, with $\mu\beta$ labels to their right.

Diagram 7: A circle with two vertices on the left and right sides. The top vertex is labeled μ . The bottom vertex is labeled α . There is a black dot on the bottom line.

Tricky interactions

$$K\underline{R}^* \left(\text{Diagram 1} \right) = -K\bar{R}^* \left(\text{Diagram 2} \right) + \left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right)$$

Diagram 1: A circle with a horizontal line through its center. The top and bottom vertices of the circle are labeled μ . There are two dots on the bottom horizontal line.

Diagram 2: A circle with a horizontal line through its center. The top and bottom vertices of the circle are labeled μ . There are two dots on the bottom horizontal line.

Diagram 3: A vertical line with two open circles at the top and bottom. A dot is on the line between the two open circles, and it is labeled μ .

Diagram 4: A circle with two vertices on the left and right sides. The top vertex is labeled μ .

- Higher order UV may create vectors in remaining diagram
- IR Taylor should be sensitive to those!

$$\left(\text{Diagram 3} \right) K \left(\text{Diagram 4} \right) = -2 \left(\text{Diagram 3} \right) K \left(\text{Diagram 5} \right) p^\alpha \quad (\text{UV Taylor})$$

Diagram 5: A circle with two vertices on the left and right sides. The top vertex is labeled μ . The bottom vertex is labeled α . There is a dot on the bottom horizontal line.


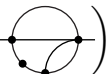
$$= -2g^{\alpha\beta} \left(\text{Diagram 3} \right) K \left(\text{Diagram 6} \right) \quad (\text{IR Taylor})$$

Diagram 6: A circle with two vertices on the left and right sides. The top vertex is labeled μ . The bottom vertex is labeled α . There is a dot on the bottom horizontal line, and it is labeled $\mu\beta$.



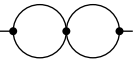
$$= -2D \left(\frac{1}{D} \text{Diagram 3} \right) K \left(\frac{1}{D} \text{Diagram 6} \right) \quad (\text{Projection})$$

Diagram 7: A circle with two vertices on the left and right sides. The top vertex is labeled μ . The bottom vertex is labeled μ . There is a dot on the bottom horizontal line.

Complete example (I)

$$K\bar{R}^* \left(\text{Diagram 1} \right) = K\bar{R}^* \left(\text{Diagram 2} \right)$$



UV subdiagrams:

$$\left\{ \{ \emptyset \}, \left\{ \text{Diagram 3} \right\}, \left\{ \text{Diagram 4} \right\}, \left\{ \text{Diagram 5} \right\} \right\}$$




Complete example (I)

$$K\bar{R}^* \left(\text{Diagram 1} \right) = K\bar{R}^* \left(\text{Diagram 2} \right)$$

UV subdiagrams:

$$\left\{ \{ \emptyset \}, \{ \text{Diagram 3} \}, \{ \text{Diagram 4} \}, \{ \text{Diagram 5} \} \right\}$$

IR subdiagrams:

$$\left\{ \{ \emptyset \}, \{ \text{Diagram 6} \}, \{ \text{Diagram 7} \} \right\}$$

Complete example (II)

$$\begin{aligned}
K\bar{R}^* \left(\text{Diagram 1} \right) &= K \left(\text{Diagram 2} \right) - K \left(\text{Diagram 3} \right) \text{Diagram 4} \\
&+ K\underline{R}^* \left(\text{Diagram 5} \right) K \left(\text{Diagram 6} \right) \text{Diagram 7} \\
&- K\underline{R}^* \left(\text{Diagram 8} \right) \text{Diagram 9} \\
&+ K\underline{R}^* \left(\text{Diagram 10} \right) K \left(\text{Diagram 11} \right) \\
&+ K\underline{R}^* \left(\text{Diagram 12} \right) K\bar{R} \left(\text{Diagram 13} \right)
\end{aligned}$$

Results

Current status

Four loop beta function recomputed with R^* in 4 hours

Goals:

- 5 loop beta function with generic colour group
- Moments of 4-loop splitting functions from 3-loop integrals
- Generalise R^* to massless external states (?)

Conclusion

- Calculated up to $N = 11$ for γ_{NS}
- Calculated n_f^0 in large n_c limit of $\gamma_{NS} \rightarrow$ new term in cusp anomalous dimension
- Constructed an R^* method that works for QCD
- Written efficient R^* +Forcer package that in principle could compute superficial divergences of any 5-loop diagram with massive external lines

Acknowledgements

This work is supported by the ERC Advanced Grant no. 320651,
“HEPGAME”