A Guided Tour through an NNLO Subtraction Scheme

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LHC and precision

- Discovery of Higgs boson and absence of enduring evidence for new physics at LHC → **precision physics programme** (NNLO and beyond).

What can precision physics do for me?

- Extensive studies of Higgs boson: fully understand the nature of EWSB.
- Search for BSM physics through subtle deviations from SM background.
- Determine fundamental parameters of nature.

LHC and precision

- ~ 1% level theoretical precision for a hadron collider extremely challenging!

What can I do for precision physics?

Two main difficulties in NNLO calculations:

1) Evaluating amplitudes with two (or more) loops.
   - Extensive effort on several different fronts.

2) Handling complicated IR divergences from two-parton emissions.

Infrared singularities

Higher order corrections contain:

- **Real corrections**
  - Unresolved phase space: emitted particle is soft and/or collinear.
  - IR singularities **after integration.**
- **Virtual** corrections
  - Born-like phase space.
  - **Explicit** IR singularities from loop integration..
- **Collinear subtractions**
  - Redefinition of pdfs to absorb collinear singularities
  - **Explicit** IR singularities.

Goal: Extract and cancel all singularities *prior* to integration.

Solved at NLO (Catani-Seymour, Frixione-Kunszt-Signer, ...).

IR singularities at NNLO

– Slicing:
$$\int |\mathcal{M}|^2 F_J d\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 F_J d\phi_d\right]_{\text{s.c.}} + \int_\delta^1 |\mathcal{M}_J|^2 F_J d\phi_4 + \mathcal{O}(\delta)$$

Born-like NLO+jet

- qT (Catani, Grazzini ‘07): VV, VH, HH
- N-jettiness (Gaunt *et al* ‘15; Boughezal *et al* ‘15): digamma, Vj, Hj, VH, single top

– Subtraction:
$$\int |\mathcal{M}|^2 F_J d\phi_d = \int \left(|\mathcal{M}_J|^2 F_J - S\right) d\phi_4 + \int S d\phi_d$$

- Antenna (Gehrmann-de Ridder, Gehrmann, Glover ‘05, …): dijet, Hj, Vj
- FKS+sector decomposition (Czakon ‘10, ‘11; Boughezal, Petriello, Melnikov ‘12): ttbar, Hj, single top
- Projection-to-Born (Cacciari *et al* ‘15): VBF, single top
- CoLoRFulNNLO (Somogyi, Trocsanyi, Del Duca ‘05, ...): e+e-

The NNLO Revolution

- Slicing & subtraction tools highly successful: all $2 \to 2$ processes known at NNLO.
- BUT: room for improvement. Current subtraction schemes:
  - Are complicated.
  - Obscure the origin of the singularities.
  - Are process-dependent.
- Ideally:
  - Local.
  - Straightforward with clear origin of singularities.
  - Explicit (if possible, analytic) cancellation of poles.
  - Process-independent.
  - Allowing four-dimensional evaluation.

Residue-Improved Sector Decomposition

Residue-improved sector decomposition:

- [Czakon '10, '11; Boughezal, Melnikov, Petriello '12]
- Modification allowing four-dimensional calculation of matrix elements [Czakon, Heymes '14].
- Same strategy as FKS, + *sectors* to separate overlapping singularities.
- Expect simplification when recombining (as in FKS) – **not apparent** in above formulations.
- I will present a different subtraction scheme based on these ideas:
  - Focus on gauge-invariant matrix elements → independent treatment of soft and collinear singularities.
  - Easier recombination of sectors → explicit pole cancellation for different kinematic structures.

Outline

- **NLO:**
  - Notation for limits.
  - Subtraction procedure.
  - Combining real, virtual and collinear subtraction contributions and cancelling poles.
- **NNLO:**
  - Real-real contribution: extension of NLO ideas:
    - Separation of soft and collinear regularization.
    - Partitions and sectors.
  - Combining contributions and pole cancellation.
- Proof-of-principle calculation and comparison.

Outline (II)

Focus on $pp \to V + X$:

- Color singlet final state (Drell-Yann, diboson, *HV*).
- Extension to colored final states is conceptually straightforward, somewhat complicated in practice.
- Color singlet decay (e.g. $H \to bb$) also calculated.

In this talk, consider partonic channel $q\bar{q} \to V + n\ g$:

- Most complicated singular structure; other partonic channels are simplifications of this.
- All partonic channels relevant for DY production calculated.

Subtraction at NLO

$$\mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{R}}. + \mathrm{d}\sigma^{\mathrm{CV}}$$

Focus on real radiation – process $q\bar{q} \to V + g$:

$$\mathrm{d}\sigma^{\mathrm{R}} = \frac{1}{2s} \int [\mathrm{d}g_4] F_{LM}(1,2,4) \equiv \langle F_{LM}(1,2,4) \rangle.$$

$$F_{LM}(1,2,4) = \mathrm{dLips}_V \, |\mathcal{M}(1,2,4,V)|^2 \, \mathcal{F}_{\mathrm{kin}}(1,2,4,V).$$

$$[\mathrm{d}g_4] = \frac{\mathrm{d}^{d-1} p_4}{(2\pi)^d 2E_4} \theta(E_{\max} - E_4)$$

Lorentz-inv. Phase space for V **(incl. delta-fn)**

Matrix-element sq.

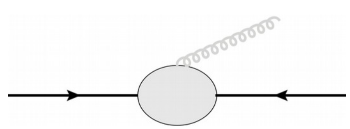
IR-safe observable

Integration in partonic CoM frame

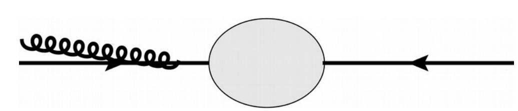
Arbitrarily large energy parameter

Singular regions:

- g_4 → soft.



- g_4 → collinear to either initial state parton.



Iterative subtraction

Define operators:

$$S_i A = \lim_{E_i \to 0} A \qquad C_{ij} A = \lim_{\rho_{ij} \to 0} A \qquad \rho_{ij} = 1 - \cos\theta_{ij}$$

Rewrite as

$$\begin{aligned}\langle F_{LM}(1,2,4)\rangle = & \langle S_4 F_{LM}(1,2,4)\rangle + \\ & \langle (C_{41} + C_{42})(I - S_4) F_{LM}(1,2,4)\rangle + \\ & \langle (I - C_{41} - C_{42})(I - S_4) F_{LM}(1,2,4)\rangle .\end{aligned}$$

- First term: soft gluon decouples completely → need upper bound: $E_{\max}$.
- Second term: collinear and soft+collinear gluon decouples partially or completely.
- Singularities made explicit by **integrating over decoupled gluon.**
- Third term: finite, can be integrated numerically in 4-dimensions.

Soft and soft+collinear limits

Soft limit: eikonal function

$$S_4 F_{\mathrm{LM}}(1,2,4) = g_{s,b}^2 \mathrm{Eik}(1,2,4) F_{LM}(1,2) = g_{s,b}^2 2C_F \frac{p_1 \cdot p_2}{(p_1 \cdot p_4)(p_2 \cdot p_4)} F_{LM}(1,2)$$
$$= g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{\rho_{12}}{\rho_{14}\rho_{24}} F_{LM}(1,2).$$

Gluon decouples from F_{LM}, so collinear limits act on prefactor only:

$$C_{41} S_4 F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{1}{\rho_{14}} F_{LM}(1,2),$$
$$C_{42} S_4 F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{1}{\rho_{24}} F_{LM}(1,2).$$

$$\Rightarrow S_4 (I - C_{41} - C_{42}) F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \left(\frac{\rho_{12}}{\rho_{14}\rho_{24}} - \frac{1}{\rho_{14}} - \frac{1}{\rho_{24}} \right) F_{LM}(1,2) = 0.$$

using $\rho_{12} = 2, \ \rho_{24} = 2 - \rho_{14}$

(for color singlet final state).

We only need to consider **collinear** limits $\langle (C_{41} + C_{42}) F_{LM}(1,2,4) \rangle$.

Collinear limits

The collinear limit is

$$C_{41}F_{LM}(1,2,4) = \frac{g_{s,b}^2}{E_4^2\rho_{41}}(1-z)P_{qq}(z)\left(\frac{F_{LM}(z\cdot 1,2)}{z}\right).$$

$$z = 1 - E_4/E_1$$

$$P_{qq}(z) = P_{qq}^{(0)}(z) + \mathcal{O}(\epsilon)$$

Flux factor

Gluonic *angles* decouple:

$$\int_{-1}^{1} \mathrm{d}\cos\theta_{14}\,(\sin\theta_{14})^{d-4}\,\frac{1}{\rho_{14}} = 2^{-2\epsilon}\int_0^1 \mathrm{d}\eta_{14}(\eta_{14})^{-1-\epsilon}(1-\eta_{41})^{-\epsilon} = -\frac{2^{-2\epsilon}}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}.$$

$$\int \frac{\mathrm{d}\Omega_4^{(d-2)}}{(2\pi)^{d-1}} g_{s,b}^2 = \frac{\alpha_s(\mu)}{2\pi}\frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \equiv [\alpha_s].$$

$$\eta_{ij} = \rho_{ij}/2$$

and changing $\mathrm{d}E_4 \to \mathrm{d}z$:

$$\langle C_{41}F_{LM}(1,2,4)\rangle = -[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int_{z_{\min}}^{1}\frac{\mathrm{d}z}{(1-z)^{2\epsilon}}P_{qq}(z)\frac{F_{LM}(z\cdot 1,2)}{z}.$$

$$\mathrm{z}_{\min} = 1 - E_{\max}/E_1$$

But need enough energy to produce *V* : can extend integration region to 0.

Recasting collinear limit

$$\langle C_{41} F_{LM}(1,2,4)\rangle = -[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z)\frac{F_{LM}(z\cdot 1,2)}{z}.$$

- Collinear pole explicit.
- Soft pole z → 1 regulated.
- Convolution of splitting function with matrix element:
  - Should cancel against collinear subtraction.
  - Rewrite in terms of **plus-prescription** with **new splitting function**

$$\mathcal{P}_{qq,R}(z) = \hat{P}_{qq}^{(0)}(z) + \epsilon \mathcal{P}_{qq,R}^{(\epsilon)}(z)$$

$$\Rightarrow \langle C_{41} F_{LM}(1,2,4)\rangle = -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \Bigg[-\left(\frac{C_F}{\epsilon}+\frac{3C_F}{2}\right)\langle F_{LM}(1,2)\rangle + \int_0^1 \mathrm{d}z \mathcal{P}_{qq,R}(z)\left\langle \frac{F_{LM}(z\cdot 1,2)}{z}\right\rangle\Bigg].$$

Combining contributions & cancelling poles

$$2s\cdot \mathrm{d}\sigma^{\mathrm{R}} = 2[\alpha_s]s^{-\epsilon}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\langle F_{LM}(1,2)\rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4)\rangle$$

$$-\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\mathcal{P}_{qq,R}(z)\left\langle \frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle.$$

$$\hat{O}_{\mathrm{NLO}} \equiv (I-C_{41}-C_{42})(I-S_4)$$

- Poles in first term cancel with virtual (extracted using [Catani '98]):

$$2s\cdot \mathrm{d}\sigma^{V} = -2[\alpha_s]\cos(\pi\epsilon)\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)s^{-\epsilon}\langle F_{LM}(1,2)\rangle + \left\langle F_{LV}^{\mathrm{fin}}(1,2)\right\rangle.$$

- Poles in third term cancel with collinear subtraction:

$$2s\cdot \mathrm{d}\sigma^{\mathrm{CV}} = \frac{\alpha_s(\mu)}{2\pi\epsilon}\int_0^1 \mathrm{d}z\, \hat{P}_{qq}^{(0)}(z)\left\langle \frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle.$$

- Cancellation occurs *within each structure!*

Evaluation in four dimensions

After cancelling poles, we can take the $\epsilon \to 0$ limit and compute everything in four dimensions.

$$2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \left\langle F_{LV}^{\mathrm{fin}}(1,2) + \frac{\alpha_s(\mu)}{2\pi} \left[\frac{2}{3}\pi^2 C_F F_{LM}(1,2) \right] \right\rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4) \rangle +$$
$$+ \frac{\alpha_s(\mu)}{2\pi} \int_0^1 \mathrm{d}z \left[\ln\frac{s}{\mu^2} \hat{P}_{qq}^{(0)}(z) - \mathcal{P}_{qq,R}^{(\epsilon)}(z) \right] \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle .$$

Sum of:

- Lower particle-multiplicity terms, with or without convolutions with splitting functions.
- Real emission term, with singular configurations removed by iterated subtraction.
- Finite remainder of virtual corrections.

NNLO calculation

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

- Double virtual: two-loop corrections
  - Poles following [Catani '98].
  - Two-loop finite and one-loop finite squared.
- Real-virtual: one-loop, one emission
  - Same kinematics as at NLO – same subtraction strategy (but more complicated).
  - Lower multiplicity terms convoluted with splitting functions.
  - Terms with singular limits removed through iterative subtraction.
- Convolutions with A-P splitting functions

NNLO: Real-real Corrections

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

Real-real corrections – process $q\bar{q} \to V + gg.$

$$2s \cdot \mathrm{d}\sigma^{\mathrm{RR}} = \frac{1}{2!}\int [\mathrm{d}g_4][\mathrm{d}g_5] F_{LM}(1,2,4,5).$$

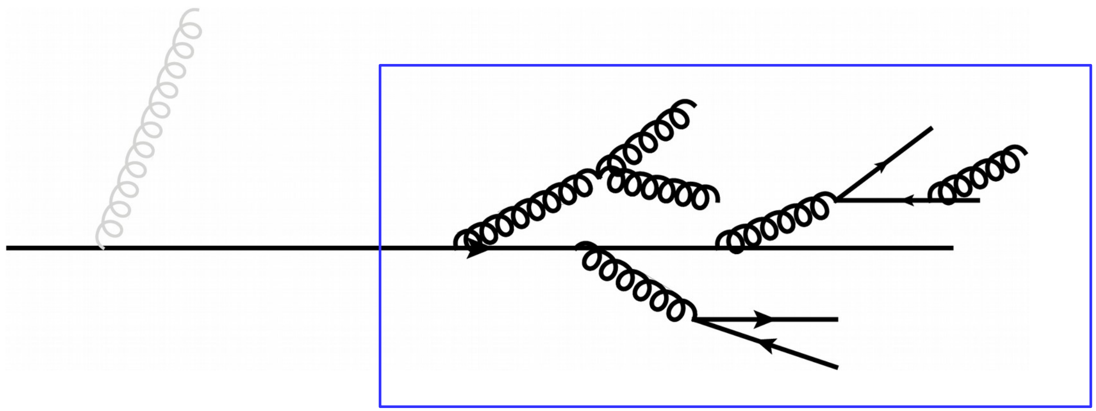
Singularity structure much more complicated:

- g_4 or g_5 → soft.
- g_4 or g_5 → collinear to initial state partons.
- g_4 or g_5 → collinear to each other.
- Combination of the above – can approach **each limit in different ways!**

Separating the singularities is the name of the game!

Color coherence

- On-shell, gauge-invariant QCD scattering amplitudes : color coherence.
- Soft gluon cannot resolve details of later splittings; only sees total color charge.



➡ Soft emissions can be treated **independently** of collinear emissions:

- Regularize soft singularities first, then collinear singularities.
- Energies and angles can be independently parametrized – no need for energy-angle ordering.

Treatment of real-real singularities

- **Step 1:** New limit operators.

$$\mathbb{S}A = \lim_{E_4,E_5\to 0} A, \text{ at fixed } E_5/E_4,$$

$$\mathbb{C}_i A = \lim_{\rho_{4i},\rho_{5i}\to 0} A, \text{ with non vanishing } \rho_{4i}/\rho_{5i}, \rho_{45}/\rho_{4i}, \rho_{45}/\rho_{5i},$$

and recall $S_i A = \lim_{E_i\to 0} A \qquad C_{ij} A = \lim_{\rho_{ij}\to 0} A.$

- **Step 2:** Order gluon energies $E_4 > E_5$.

$$2\,\mathrm{s}\cdot \mathrm{d}\sigma^{\mathrm{RR}} = \int [\mathrm{d}g_4][\mathrm{d}g_5]\theta(E_4 - E_5) F_{LM}(1,2,4) \equiv \langle F_{LM}(1,2,4,5)\rangle.$$

- Gluon energies bounded by $E_{\max}$.
- Energies defined in CoM frame.
- Soft singularities: either double soft or g_5 soft.

Soft singularities

- **Step 3:** Regulate the soft singularities:

$$\langle F_{LM}(1,2,4,5)\rangle = \langle \mathbb{S} F_{LM}(1,2,4,5)\rangle + \langle S_5(I-\mathbb{S})F_{LM}(1,2,4,5)\rangle$$
$$+\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\rangle.$$

- First term: both g_4 and g_5 soft.
- Second term: g_5 soft, soft singularities in g_4 are regulated.
- Third term: regulated against all soft singularities.
- All three terms contain **(overlapping)** collinear singularities.

Phase-space partitioning

- **Step 4:** Introduce **phase-space partitions**

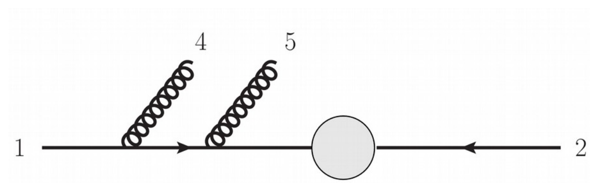
$$1 = w^{14,15} + w^{24,25} + w^{14,25} + w^{15,24}.$$

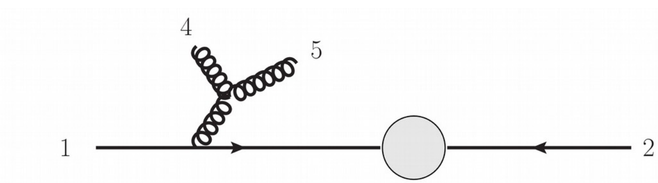
with

$$C_{42} w^{14,15} = C_{52} w^{14,15} = 0 \quad \Rightarrow \quad w^{14,15} \text{ contains } C_{41},\ C_{51},\ C_{45}$$

$$C_{41} w^{24,25} = C_{51} w^{24,25} = 0 \qquad\qquad w^{24,25} \text{ contains } C_{42},\ C_{52},\ C_{45}$$

Triple collinear
partition



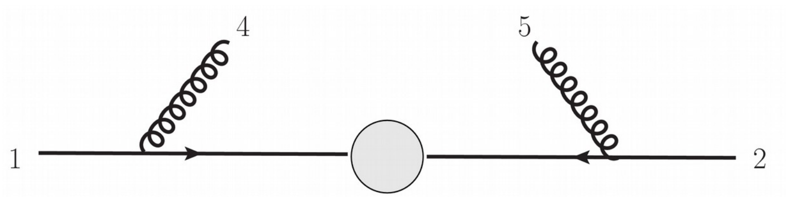


and

$$C_{42} w^{14,25} = C_{51} w^{14,25} = C_{45} w^{14,25} = 0 \quad \Rightarrow \quad w^{14,25} \text{ contains } C_{41},\ C_{52}$$

$$C_{41} w^{15,24} = C_{52} w^{15,24} = C_{45} w^{15,24} = 0 \qquad\qquad w^{15,24} \text{ contains } C_{42},\ C_{51}$$

Double collinear
partition



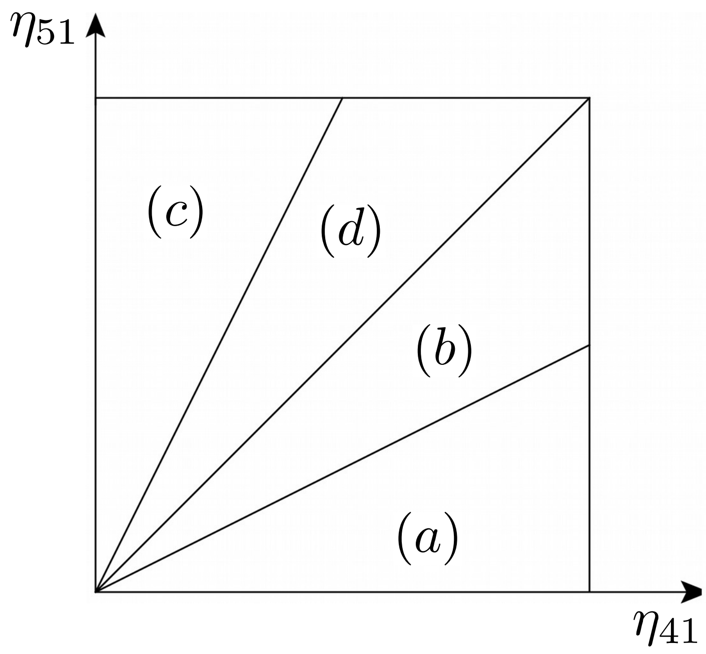
Sector Decomposition

- **Step 5: Sector decomposition:**

- Triple collinear sectors still have **overlapping** singularities.
- Define angular ordering to separate singularities.

$$\eta_{ij} = \rho_{ij}/2$$

$$1 = \theta\left(\eta_{51} < \frac{\eta_{41}}{2}\right) + \theta\left(\frac{\eta_{41}}{2} < \eta_{51} < \eta_{41}\right) + \theta\left(\eta_{41} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{41} < \eta_{51}\right) \equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}.$$



- Thus the limits are

$$\theta^{(a)} : C_{51} \qquad \theta^{(b)} : C_{45}$$

$$\theta^{(c)} : C_{41} \qquad \theta^{(d)} : C_{45}$$

Sectors *a,c* and *b,d* same to *4* ↔ *5*, but recall energy ordering.

Removing collinear singularities

Then we can write soft regulated term as

$$\left\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\right\rangle = \left\langle F_{LM}^{s_r c_s}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_t}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_r}(1,2,4,5)\right\rangle,$$

with

$$\langle F_{LM}^{s_r c_r}(1,2,4,5)\rangle = \sum_{(ij)\in dc} \left\langle [I-\mathbb{S}]\,[I-S_5]\left[(I-C_{5j})(I-C_{4i})\right][\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,j5}F_{LM}(1,2,4,5)\right\rangle$$
$$+\sum_{i\in tc}\Bigg\langle [I-\mathbb{S}]\,[I-S_5]\Big[\theta^{(a)}\,[I-\mathbb{C}_i]\,[I-C_{5i}] + \theta^{(b)}\,[I-\mathbb{C}_i]\,[I-C_{45}]$$
$$+\,\theta^{(c)}\,[I-\mathbb{C}_i]\,[I-C_{4i}] + \theta^{(d)}\,[I-\mathbb{C}_i]\,[I-C_{45}]\Big][\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,i5}$$
$$\times\, F_{LM}(1,2,4,5)\Bigg\rangle.$$

- All singularities removed through iterated subtractions – evaluated in 4-dimensions.
- Only term involving fully-resolved matrix element $F_{LM}(1,2,4,5)$.

Removing collinear singularities

$$\left\langle (I - S_5)(I - \mathbb{S}) F_{LM}(1,2,4,5) \right\rangle = \left\langle F_{LM}^{s_r c_s}(1,2,4,5) \right\rangle + \left\langle F_{LM}^{s_r c_t}(1,2,4,5) \right\rangle + \left\langle F_{LM}^{s_r c_r}(1,2,4,5) \right\rangle .$$

Remaining two terms contain singularities:

$\left\langle F_{LM}^{s_r c_s}(1,2,4,5) \right\rangle$

- Soft-regulated single-collinear subtraction.
- Partitioning factors and sectors: **one collinear singularity** in each term.

$\left\langle F_{LM}^{s_r c_t}(1,2,4,5) \right\rangle$

- Triple-collinear subtraction – all other singularities regulated.

Treating singular limits

We have four singular subtraction terms:

$$\langle \mathbb{S} F_{LM}(1,2,4,5)\rangle \quad \langle S_5(I-\mathbb{S})F_{LM}(1,2,4,5)\rangle \quad \langle F_{LM}^{s_r c_s}(1,2,4,5)\rangle \quad \langle F_{LM}^{s_r c_t}(1,2,4,5)\rangle$$

We know how to treat them:

- Gluon(s) decouple partially or completely.
- Decouple completely:
  - Integrate over gluonic angles and energy.
- Decouple partially:
  - Integrate over gluonic angles.
  - Integral(s) over energy → integrals over splitting function in *z.*
- Results in **lower particle multiplicity terms** convoluted with (new) splitting functions.

Soft subtraction terms

Double soft subtraction: $\langle S\!\!\!S F_{LM}(1,2,4,5)\rangle$

- Both gluons decouple: $S\!\!\!S F_{LM}(1,2,4,5) = g_{s,b}^4 \text{Eik}(1,2,4,5) F_{LM}(1,2).$

Double eikonal function

- Overall energy factorizes → integrand *independent* of partonic energy.
- Integral is **constant** for color-singlet production.
- Abelian contribution: product of NLO structures.
- Non-abelian: more complicated.
- Integrate over relative energies and over gluonic angles *numerically.* (Analytic evaluation not particularly complicated).

$$\langle S\!\!\!S F_{LM}(1,2,4,5)\rangle = [\alpha_s]^2 \langle E_{\max}^{-4\epsilon} F_{LM}(1,2)\rangle \left(\frac{c_4}{\epsilon^4} + \frac{c_3}{\epsilon^3} + \frac{c_2}{\epsilon^2} + \frac{c_1}{\epsilon} + c_0\right).$$

c_i : Constants for color-singlet production

Single-soft subtraction term

Single-soft subtraction: $\langle (I-\mathbb{S}) S_5 F_{LM}(1,2,4,5) \rangle$

- **Gluon 5** decouples – integrate over it:

$$\langle (I-\mathbb{S}) S_5 F_{LM}(1,2,4,5) \rangle = \frac{[\alpha_s]}{\epsilon^2} \langle E_4^{-2\epsilon} f(\rho_{12},\rho_{14},\rho_{24},\epsilon)\ (I-S_4) F_{LM}(1,2,4) \rangle.$$

- **This term is not integrable:** contains NLO-like collinear singularities.
- Regulate the singularities as done at NLO:

$$\Big\langle \big[I-\mathbb{S}\big] S_5 F_{LM}(1,2,4,5) \Big\rangle = \frac{[\alpha_s]}{\epsilon^2} \Big\langle E_4^{-2\epsilon} f(\rho_{12},\rho_{14},\rho_{24},\epsilon)$$

$$\times \big[I - C_{41} - C_{42}\big]\big[I - S_4\big] F_{LM}(1,2,4) \Big\rangle$$

$$- \frac{[\alpha_s]^2 s^{-2\epsilon}}{\epsilon^3} f(\epsilon) \int_0^1 \mathrm{d}z\, \mathcal{P}_{qq,RR_1}(z) \left\langle \frac{F_{LM}(z\cdot 1, 2) + F_{LM}(1, z\cdot 2)}{z} \right\rangle.$$

- First term: Regulated through iterative subtractions.
- Second term: LO matrix element convoluted with splitting function.

Collinear subtractions

General structure: splitting functions with **explicit poles** convoluted with lower multiplicity terms:

$$F_{LM}(z\cdot 1,2,4) \qquad F_{LM}(1,z\cdot 2,4) \qquad F_{LM}(1,2,4)$$

$$F_{LM}(z\cdot 1,2) \qquad F_{LM}(1,z\cdot 2) \qquad F_{LM}(1,2)$$

Further singularities regulated →

$$\left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(z\cdot 1,2,4)\right\rangle \qquad \left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(1,z\cdot 2,4)\right\rangle \qquad \left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(1,2,4)\right\rangle$$

$$\langle F_{LM}(z\cdot 1,\bar{z}\cdot 2)\rangle \qquad \langle F_{LM}(z\cdot 1,2)\rangle \qquad \langle F_{LM}(1,z\cdot 2)\rangle \qquad \langle F_{LM}(1,2)\rangle$$

- At NLO, pole cancellation achieved in *each structure.*
- Recombine structures from different sectors/partitions.

Double-collinear partition

In single-collinear subtraction:

$$DC = \left\langle [I - \mathbb{S}]\,[I - S_5] \left[\left(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5] \right) w^{14,25} + \left(C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5] \right) w^{24,15} \right] \times F_{LM}(1,2,4,5) \right\rangle .$$

Limit acts on phase space!

Consider one term: $\langle [I - \mathbb{S}]\,[I - S_5]\, C_{41}[\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5) \rangle$:

- Write $C_{41} w^{14,25} = \tilde{w}^{14,25}_{4||1}$ – independent of gluon 4.
- Soft & collinear limits commute:

$$\mathbb{S}(I - S_5) C_{41} F_{LM}(1,2,4,5) \sim \mathbb{S} F_{LM}(1-4,2,5) - \mathbb{S} S_5 F_{LM}(1-4,2,5) = 0.$$

- Limit reduces to $C_{41}(I - S_5) F_{LM}(1,2,4,5) = \dfrac{g^2_{s,b}}{E_4^2 \rho_{41}} \mathcal{P}_{qq}(z)\,[I - S_5]\, F_{LM}(z \cdot 1, 2, 5).$

Double-collinear partition

Limit acting on phase space: $C_{41} \int_0^1 \mathrm{d}\eta_{14}\,(\eta_{14}(1-\eta_{41}))^{-\epsilon} \to \int_0^1 \mathrm{d}\eta_{14}(\eta_{14})^{-\epsilon}$

Angular integration is then: $2^{(1-2\epsilon)} \int_0^1 \mathrm{d}\eta_{14}\eta_{14}^{-\epsilon}\frac{1}{2\eta_{41}} = -\frac{2^{-2\epsilon}}{\epsilon}.$

Limits of integration: $z = 1 - E_4/E_1$ and $E_4 > E_5 \Rightarrow z < 1 - E_5/E_1 \equiv z_{\max}(E_5).$

Find:

$$\left\langle [I - \mathbb{S}]\,[I - S_5]\,C_{41}[\mathrm{d}g_4]w^{14,25}F_{LM}(1,2,4,5)\right\rangle$$

$$= -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\int_{z_{\min}}^{z_{\max}(E_5)}\frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}}\mathcal{P}_{qq}(z)\left\langle \tilde{w}_{4||1}^{14,25}\,[I - S_5]\,F_{LM}(z\cdot 1,2,5)\right\rangle.$$

- Lower limit can be extended to 0, but upper limit cannot be extended to 1.

Double-collinear partition

$$DC = \left\langle [I - \mathbb{S}]\,[I - S_5]\left[(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5])\,w^{14,25} + (C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5])\,w^{24,15}\right] \times F_{LM}(1,2,4,5)\right\rangle.$$

Now consider the term $\langle [I - \mathbb{S}]\,[I - S_5] C_{51}[\mathrm{d}g_5] w^{24,15} F_{LM}(1,2,4,5)\rangle$:

- Proceeds analogously, but with:
  - $z = 1 - E_5/E_1$ and $E_4 > E_5 \Rightarrow z > 1 - E_4/E_1 \equiv z_{\min}(E_4)$.
  - Collinear and soft limit *both* act on gluon 5

    → different splitting function to include soft subtraction.

Get

$$\langle [I - \mathbb{S}]\,[I - S_5] C_{51}[\mathrm{d}g_5] w^{24,15} F_{LM}(1,2,4,5)\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int_{z_{\min}(E_4)}^{1} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \hat{\mathcal{P}}_{qq}^{(-)}(z) \langle \tilde{w}_{5||1}^{24,15} F_{LM}(z \cdot 1, 2, 4)\rangle.$$

$$\hat{\mathcal{P}}_{qq}^{(-)} f(z) \equiv \mathcal{P}_{qq}(z) f(z) - 2C_F f(1)$$

Combining partitions

Take first double collinear term and **rename** the resolved gluon 4:

$$\left\langle \left[I - \mathbb{S}\right] \left[I - S_5\right] C_{41} [\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5) \right\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int_0^{z_{\max}(E_4)} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \mathcal{P}_{qq}(z) \left\langle \tilde{w}_{5||1}^{15,24} \left[I - S_4\right] F_{LM}(z \cdot 1, 2, 4) \right\rangle.$$

Using $z_{\max}(E_4) = 1 - E_4/E_1 = z_{\min}(E_4)$ combine with second term:

$$\left\langle \left[I - \mathbb{S}\right] \left[I - S_5\right] \left[C_{41} [\mathrm{d}g_4] w^{14,25} + C_{51} [\mathrm{d}g_4] w^{15,24} F_{LM}(1,2,4,5) \right\rangle\right.$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int_0^1 \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \left\langle \tilde{w}_{5||1}^{15,24} \left(\hat{\mathcal{P}}_{qq}^{(-)}(z) \left[I - S_4\right] F_{LM}(z \cdot 1, 2, 4) + \right.\right.$$

$$\left.\left. \theta(z_4 - z) 2 C_F \left[I - S_4\right] F_{LM}(1,2,4) + \theta(z_4 - z) \hat{\mathcal{P}}_{qq}^{(-)}(z) S_4 F_{LM}(z \cdot 1, 2, 4) \right) \right\rangle.$$

Similar simplifications on combining terms from **double** & **triple** collinear partitions.

Triple-collinear subtraction term

Terms with double-unresolved collinear limits: $\left\langle F_{LM}^{s_r c_t}(1,2,4,5)\right\rangle$

- For triple collinear sectors, limits have complicated triple collinear splitting function:
  - Integration is non-trivial.
  - Expand the *integrand* in ϵ.
  - Evaluate numerically (analytic evaluation should also be possible).
  - Produces $1/\epsilon$ pole & finite term.

Double-real cross section: recap

We have now written complete double-real cross section as:

- Splitting functions convoluted with LO matrix elements – including explicit $1/\epsilon^4$ (and higher) poles.

$$\langle F_{LM}(z\cdot 1, \bar{z}\cdot 2)\rangle\,,\ \langle F_{LM}(z\cdot 1, 2)\rangle\,,\ \langle F_{LM}(1, z\cdot 2)\rangle\,,\ \langle F_{LM}(1,2)\rangle$$

- Splitting functions convoluted with NLO matrix elements, regulated by iterative subtraction – including explicit $1/\epsilon^2$ (and higher) poles.

$$\langle \mathcal{O}_{NLO}F_{LM}(z\cdot 1, 2, 4)\rangle\,,\ \langle \mathcal{O}_{NLO}F_{LM}(1, z\cdot 2, 4)\rangle\,,\ \langle \mathcal{O}_{NLO}F_{LM}(1, 2, 4)\rangle$$

- NNLO matrix elements, regulated by iterative subtraction – **finite.**
- All singularities made explicit.
- Evaluate in **four dimensions.**

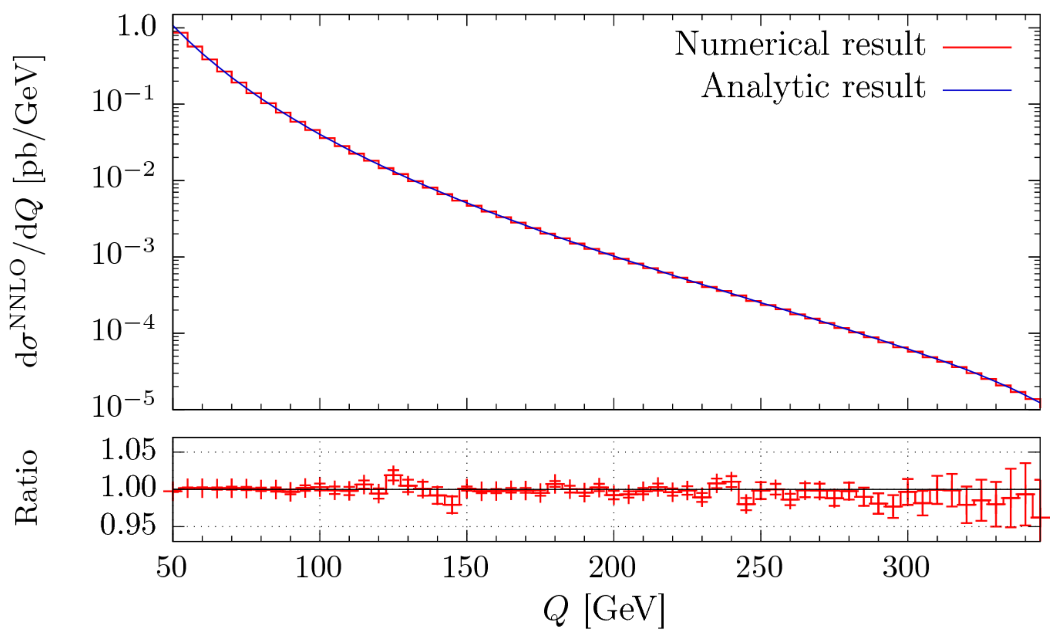
Pole cancellation

- Combine poles from real-real, real-virtual, virtual-virtual, collinear subtraction.
- Poles *must* cancel for each structure F_{LM} :
  - ✔ $\langle \mathcal{O}_{NLO} F_{LM}(z \cdot 1, 2, 4) \rangle$, $\langle \mathcal{O}_{NLO} F_{LM}(1, z \cdot 2, 4) \rangle$, $\langle \mathcal{O}_{NLO} F_{LM}(1, 2, 4) \rangle$ cancel *analytically*.
  - ✔ $\left\langle F_{LV}^{\mathrm{fin}}(1,2) \right\rangle$ cancels *analytically*.
  - ✔ $\langle F_{LM}(z \cdot 1, \bar{z} \cdot 2) \rangle$, $\langle F_{LM}(z \cdot 1, 2) \rangle$, $\langle F_{LM}(1, z \cdot 2) \rangle$, $\langle F_{LM}(1, 2) \rangle$ cancels *numerically* (double-soft and triple-collinear computed numerically).

Proof-of-principle

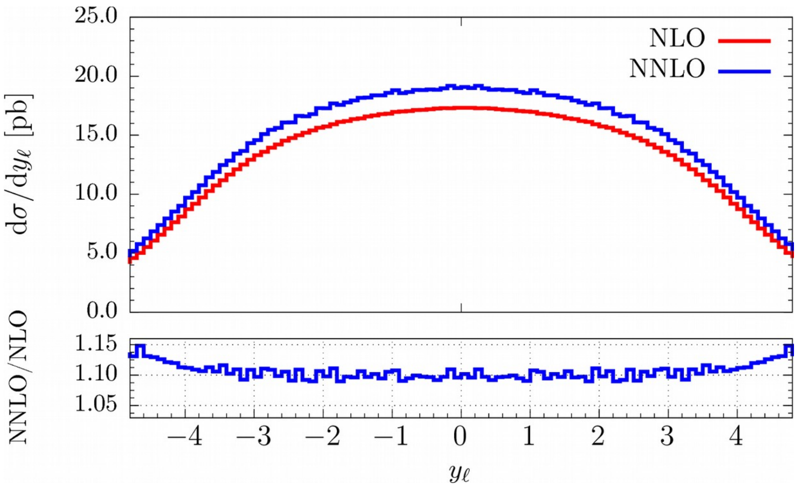
- Calculate $pp \to \gamma^* + X \to e^+e^- + X$ to NNLO (using $q\bar{q} \to \gamma^* + ng$ channel only).
- Lepton pairs with invariant mass $50 \text{ GeV} \leq Q \leq 350 \text{ GeV}$.
- Extract results from [Hamberg, Matsuura, van Neerven '91] to compare (analytic in *Q*).
- NNLO *contributions.*

$$\mathrm{d}\sigma^{\mathrm{NNLO}} = 14.471(4) \text{ pb}$$
$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{\mathrm{analytic}} = 14.470 \text{ pb}$$

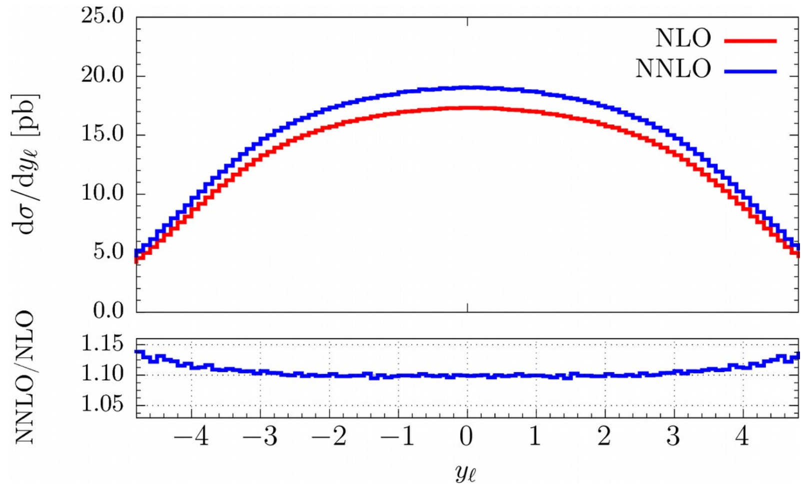


- Sub per-mille agreement in cross sections.
- Per-mille to percent agreement across **5 orders of magnitude** in *Q*.

Differential distributions (I)



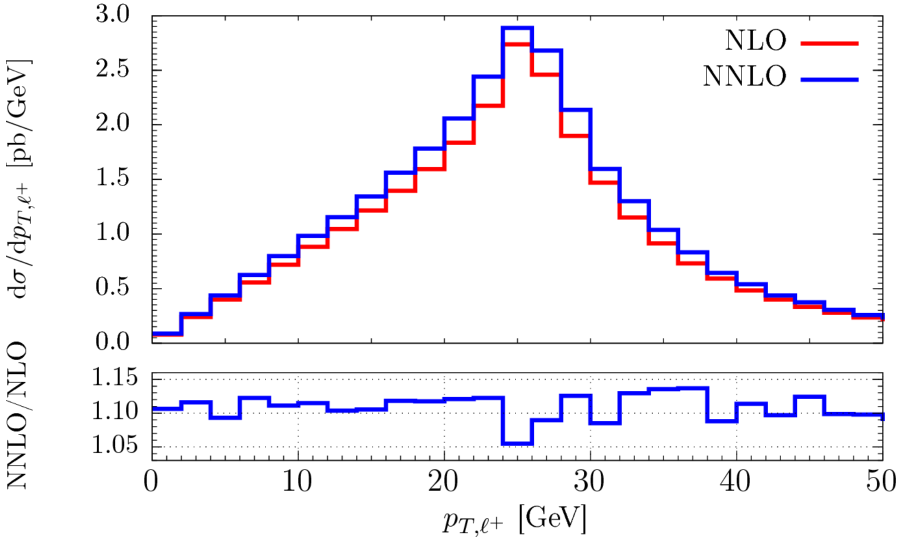
O(10 CPU hours) runtime



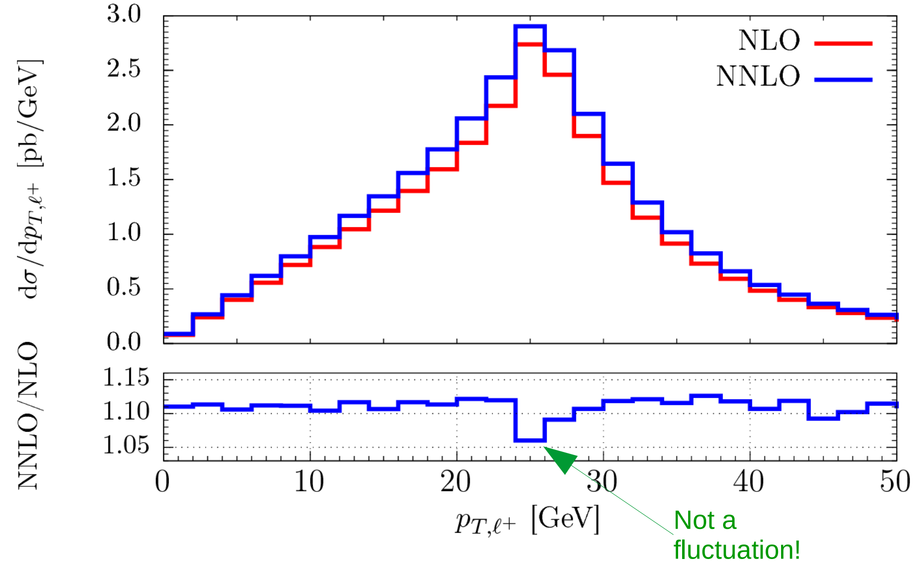
O(100 CPU hours) runtime

- Lepton rapidity.
- O(10 CPU hours): percent-level bin-to-bin fluctuations.
- O(100 CPU hours): per-mille bin-to-bin fluctuations.

Differential distributions (II)



O(10 CPU hours) runtime



O(100 CPU hours) runtime

- Lepton transverse momentum.
- O(100 CPU hours): percent-level bin-to-bin fluctuations.
- Delicate observable: receives contributions from large range of invariant masses.
  - Improves once introduce Z boson propagator.
  - Competitive with state-of-the-art NNLO codes.

Other partonic channels

- Other partonic channels (*qg, gq, gg, qq→qq)* follow same strategy, but fewer limits.
- All calculated within this approach for DY and W production.
  - Similar agreement with analytic results, including for numerically tiny channels.
- Extension to different color singlets requires only:
  - Amplitudes.
  - Phase space for color singlet.

Summary

- New method of handling NNLO subtraction, based on FKS+sector decomposition.
- Characterized by decoupling of soft and collinear limits.
- Develop iterative subtraction procedure:
  - Manifestly regulated finite term.
  - Integrated subtraction terms: convolutions of splitting function with explicit poles with lower multiplicity processes.
  - Pole cancellation independent of matrix elements.
- **Process independent** (colorless final state).
- Tested in DY and *W* production *for all partonic channels; H → bb* decay
  - Excellent agreement with analytic results.

Future work:

- Better parametrization?
- Efficient implementation in numerical integration.
- Extension to colored final states.
- Include alpha-parameters.
- …

THANK YOU!

QUESTIONS?